MHD turbulence (in 20-30 mins)

Nic Brummell
Applied Mathematics and Statistics
University of California Santa Cruz, USA

Blatant plagiarism of new review by Tobias, Cattaneo, Boldyrev

(I have permission to distribute this review)

Introduction

- Magnetic fields are ubiquitous in cosmological objects
- Heard about hydrodynamic turbulence. What is the effect of magnetic field?
- Interaction of magnetic fields with electrically-conducting fluid gives rise to a complex system whose dynamics are distinctly different from
  - non-conducting fluid
  - magnetic field in a vacuum
- From system scale to dissipation scale is a wide range => turbulent
- Electrically-conducting fluid = plasma, but in many cases, MHD will do
- => MHD TURBULENCE
- Applications: galaxy clusters, interstellar medium, stellar interiors, laboratory experiments, … etc
- It is a BIG PROBLEM: isotropic homogeneous hydro turbulence spectrum settled in the late 1940s; still arguing about MHD spectrum!

Comparison

Similarities with hydro:

- Fundamental idea of energy transfer in turbulent cascade: injection -> inertial -> dissipation
- Objective: characterise the inertial regime

Differences (more details later):

- More vector fields to deal with! (u and B)
- Dynamos! Self-magnetisation. No hydro equivalent.
- Can divide MHD turbulence into:
  - Large-scale field does not exist = dynamo case
  - Large-scale field already exists (imposed, or are just examining small region), is strong, and we are not concerned with it's origin = guide-field case
  - (we will concentrate here on the latter <sigh>)

Differences

- In hydro, generally become progressively weaker at smaller scales until eventually unconstrained. e.g. rotation, stratification
- In MHD, there is no scale below which the fluid becomes unmagnetised!
- Very important: regular turbulence becomes more isotropic at small scales; MHD turbulence becomes progressively more anisotropic.
  - Hydro turbulence: small eddies advected by larger eddies without affecting their dynamics (Galilean invariance; can remove mean flow)
  - => hydro turbulence becomes isotropic at small scale.
  - Hydro turbulence "forgets" influence of larger scales
  - Large-scale mean field cannot be removed by Galilean transformation.
  - Large-scale field mediates energy cascade at ALL scales in the inertial regime => weak small-scale turbulent fluctuations become progressively more ANISOTROPIC (easier to shuffle field lines that to bend)
  - MHD "never forgets"!

Equations

The usual …

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} + Re^{-1} \nabla^2 \mathbf{v} + \mathbf{F},$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + Rm^{-1} \nabla^2 \mathbf{B},$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{v} = 0,$$

Conserved quantities

Energy

$$E = \frac{1}{2}\int_V \left(v^2 + B^2\right)\, d^3x,$$

Cross-helicity

$$H^c = \int_V \mathbf{v}\cdot\mathbf{B}\, d^3x,$$

Magnetic helicity

$$H = \int_V \mathbf{A}\cdot\mathbf{B}\, d^3x,$$

Notice: lim B->0, conserved quantities switch to energy and kinetic helicity => fundamental differences between hydro and MHD

(Biskamp 2003) In presence of dissipation:

Energy decays faster than magnetic and cross helicity

Energy cascades to small scales (like 3D hydro) (fluctuations in equipartition)

Magnetic helicity cascades towards large-scales (like 2D hydro)

Cross-helicity also cascades towards small-scales (but is not sign-definite, so can be amplified or destroyed locally; will lead to self-organisation [later])

Waves (incompressible)

Small perturbation analysis with mean field reveals solutions as waves that propogate on the stationary and uniformly-magetised background.

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_k \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$$

$$\mathbf{b}(\mathbf{x},t) = \mathbf{b}_k \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{x})$$

Shear Alfven waves:

Dispersion - $\omega = |k_z| v_A$

Polarisation - $\mathbf{b}_k \cdot \mathbf{k} = \mathbf{b}_k \cdot \mathbf{B}_0 = 0$

Two waves: $\mathbf{v}_k = -\mathbf{b}_k$ $\mathbf{v}_k = \mathbf{b}_k$ (Compressible: “Alfven”)

Pseudo Alfven waves:

Dispersion - $\omega = |k_z| v_A$

Polarisation - $\mathbf{b}_k \cdot \mathbf{k} = 0$ and lies in the plane of $\mathbf{k}$ and $\mathbf{B}_0$.

Two waves: $\mathbf{v}_k = -\mathbf{b}_k$ $\mathbf{v}_k = \mathbf{b}_k$ (Compressible: “Slow”)

Waves

Hydro turbulence = eddies: MHD turbulence = eddies and waves

Elsasser variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \mathbf{b}$

$$\left(\frac{\partial}{\partial t} \mp \mathbf{v}_A \cdot \nabla\right) \mathbf{z}^{\pm} + \left(\mathbf{z}^{\mp} \cdot \nabla\right) \mathbf{z}^{\pm} = -\nabla P + \frac{1}{2}(\nu + \eta)\nabla^2 \mathbf{z}^{\pm} + \frac{1}{2}(\nu - \eta)\nabla^2 \mathbf{z}^{\mp} + \mathbf{f}^{\pm}$$

For ideal, incompressible, unforced MHD equations, there exists EXACT NONLINEAR SOLN that is a NON-DISPERSIVE ALFVEN WAVE PACKET

Nonlinear interactions are collisions of counter-propogating wave packets and packets get split and distorted until dissipated.

WEAK MHD TURBULENCE

Linear $(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} \sim v_A k_{\|} b_{\lambda}$ (advection along guide field) >>

nonlinear $(\mathbf{z}^{\mp} \cdot \nabla)\,\mathbf{z}^{\pm} \sim k_{\perp} b_{\lambda}^2$ (distortion/transfer of energy)

$$k_{\|} v_A \gg k_{\perp} b_{\lambda}$$

- Mean-fluctuating interactions dominate
- Takes many interactions to distort the wave packets.
- Weak MHD turbulence is an ensemble of weakly interacting waves.

Inertial scalings

ISOTROPIC wave packets interactions (Iroshnikov 1963, Kraichnan 1965)

$k_{\parallel} \sim k_{\perp} \sim 1/\lambda$

Energy transfer requires MANY interactions $v_{\lambda} \propto b_{\lambda} \propto \lambda^{1/4}$

Leads to $$E_{IK}(k) \sim |v_k|^2 k^2 \propto k^{-3/2}$$

ANISOTROPIC wave packets interactions

Weak turbulence: (Galtier et al. 2000)

Resonant interactions dictate $k_{\perp} \gg k_{\parallel}$

Pseudo-Alfven waves decouple

Energy cascades in direction of large $k_{\perp}$

$$E(k_{\perp}) \propto k_{\perp}^{-2} \qquad v_{\lambda} \propto b_{\lambda} \propto \lambda^{1/2}$$

Inertial scalings (cont)

ANISOTROPIC wave packets interactions

Strong turbulence: (Goldreich& Sridhar 1995; Cho & Vishniac 2000; Boldyrev 2005)

Achieved when CRITICAL BALANCE $k_{\|} v_A \sim k_{\perp} b_{\lambda}$

- Fluctuating-fluctuating interactions comparable to mean-fluctuating
- Field lines strongly bent now
- Wave packets are distorted in one one interaction not many
- Small wave packets are not guided by the original guide field but the locally distorted field (disorted by larger packets)
- Scale-dependent anisotropy

Spectrum: $$E_{GS}(k_{\perp}) \propto k_{\perp}^{-5/3}$$

Inertial scalings (cont)

ANISOTROPIC wave packets interactions

Strong turbulence: (cont)

BUT numerics find $E(k_\perp) \propto k_\perp^{-3/2}$ Flatter!

Latest theories explain this: (Boldyrev 2005, 2006)

Explains longer energy transfer times at small scales by DYNAMICAL ALIGNMENT.

v and b aligned with scale dependent angle; reduces nonlinear interactions

(Balanced … unbalanced …)

Conclusions

MHD guide-field turbulence very different from hydro

Can be WEAK or STRONG:

- Weak: mean-fluctuating interactions dominate
- Strong: fluctuating-fluctuating are comparable
- Transition in the inertial cascade? If strength of fluctuating-fluctuating interactions increase with smaller scales, should be a transition from weak to strong turbulence in a sufficiently extended inertial regime (very difficult to simulate).

Energy cascade ANISOTROPIC: transfer mostly transverse to the local mean field.

Polarisation vectors tend to align in the transverse plane: consequences for energy spectrum.

Elongation and alignment increase with decreasing scale