MHD turbulence (in 20-30 mins)

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Blatant plagiarism of new review by Tobias, Cattaneo, Boldyrev
(I have permission to distribute this review)
Introduction

- Magnetic fields are ubiquitous in cosmological objects
- Heard about hydrodynamic turbulence. What is the effect of magnetic field?
- Interaction of magnetic fields with electrically-conducting fluid gives rise to a complex system whose dynamics are distinctly different from
  - non-conducting fluid
  - magnetic field in a vacuum
- From system scale to dissipation scale is a wide range => turbulent
- Electrically-conducting fluid = plasma, but in many cases, MHD will do
- => MHD TURBULENCE
- Applications: galaxy clusters, interstellar medium, stellar interiors, laboratory experiments, … etc
- It is a BIG PROBLEM: isotropic homogeneous hydro turbulence spectrum settled in the late 1940s; still arguing about MHD spectrum!
Comparison

Similarities with hydro:

- Fundamental idea of energy transfer in turbulent cascade: injection \rightarrow inertial \rightarrow dissipation
- Objective: characterise the inertial regime

Differences (more details later):

- More vector fields to deal with! (u and B)
- Dynamos! Self-magnetisation. No hydro equivalent.
- Can divide MHD turbulence into:
  - Large-scale field does not exist = dynamo case
  - Large-scale field already exists (imposed, or are just examining small region), is strong, and we are not concerned with it’s origin = guide-field case
  - (we will concentrate here on the latter <sigh>)}
Differences

- In hydro, generally become progressively weaker at smaller scales until eventually unconstrained. e.g. rotation, stratification
- In MHD, there is no scale below which the fluid becomes unmagnetised!
- Very important: regular turbulence becomes more isotropic at small scales; MHD turbulence becomes progressively more anisotropic.
  - Hydro turbulence: small eddies advected by larger eddies without affecting their dynamics (Galilean invariance; can remove mean flow)
  - => hydro turbulence becomes isotropic at small scale.
  - Hydro turbulence “forgets” influence of larger scales
  - Large-scale mean field cannot be removed by Galilean transformation.
  - Large-scale field mediates energy cascade at ALL scales in the inertial regime => weak small-scale turbulent fluctuations become progressively more ANISOTROPIC (easier to shuffle field lines that to bend)
  - MHD “never forgets”!
Equations

The usual …

\[ \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} + Re^{-1} \nabla^2 \mathbf{v} + \mathbf{F}, \]
\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + Rm^{-1} \nabla^2 \mathbf{B}, \]
\[ \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{v} = 0, \]
Conserved quantities

Energy

\[ E = \frac{1}{2} \int_V (v^2 + B^2) \, d^3x, \]

Notice: \( \lim B \to 0 \), conserved quantities switch to energy and kinetic helicity \( \Rightarrow \) fundamental differences between hydro and MHD

Cross-helicity

\[ H^c = \int_V \mathbf{v} \cdot \mathbf{B} \, d^3x, \]

Magnetic helicity

\[ H = \int_V \mathbf{A} \cdot \mathbf{B} \, d^3x, \]

(Biskamp 2003) In presence of dissipation:

Energy decays faster than magnetic and cross helicity

Energy cascades to small scales (like 3D hydro) (fluctuations in equipartition)

Magnetic helicity cascades towards large-scales (like 2D hydro)

Cross-helicity also cascades towards small-scales (but is not sign-definite, so can be amplified or destroyed locally; will lead to self-organisation [later])
Waves (incompressible)

Small perturbation analysis with mean field reveals solutions as waves that propagates on the stationary and uniformly-magnetised background.

\[ v(x, t) = v_k \exp(-i\omega t + ik \cdot x) \]
\[ b(x, t) = b_k \exp(-i\omega t + ik \cdot x) \]

**Shear Alfven waves:**

- Dispersion: \[ \omega = |k_z| v_A \]
- Polarisation: \[ b_k \cdot k = b_k \cdot B_0 = 0 \]
- Two waves: \[ v_k = -b_k \quad v_k = b_k \] (Compressible: “Alfven”)

**Pseudo Alfven waves:**

- Dispersion: \[ \omega = |k_z| v_A \]
- Polarisation: \[ b_k \cdot k = 0 \text{ and lies in the plane of } k \text{ and } B_0. \]
- Two waves: \[ v_k = -b_k \quad v_k = b_k \] (Compressible: “Slow”)
Hydro turbulence = eddies: MHD turbulence = eddies and waves

Elsasser variables: \( z^\pm = v \pm b \)

\[
\begin{aligned}
\frac{\partial}{\partial t} + (z^\mp \cdot \nabla) z^\pm &= -\nabla P + \frac{1}{2}(\nu + \eta)\nabla^2 z^\pm + \frac{1}{2}(\nu - \eta)\nabla^2 z^\mp + f^\pm
\end{aligned}
\]

For ideal, incompressible, unforced MHD equations, there exists \textbf{EXACT NONLINEAR SOLN} that is a \textbf{NON-DISPERSIVE ALFVEN WAVE PACKET}

Nonlinear interactions are collisions of counter-propogating wave packets and packets get split and distorted until dissipated.

**WEAK MHD TURBULENCE**

Linear \( (v_A \cdot \nabla) z^\pm \sim v_A k_{\parallel} b_\lambda \)  (advection along guide field) \( \gg \)

nonlinear \( (z^\mp \cdot \nabla) z^\pm \sim k_{\perp} b_\lambda^2 \)  (distortion/transfer of energy)

- Mean-fluctuating interactions dominate
- Takes many interactions to distort the wave packets.
- Weak MHD turbulence is an ensemble of weakly interacting waves.
Inertial scalings

**ISOTROPIC wave packets interactions** (Iroshnikov 1963, Kraichnan 1965)

\[ k_\parallel \sim k_\perp \sim 1/\lambda \]

Energy transfer requires MANY interactions

Leads to

\[ E_{IK}(k) \sim |v_k|^2 k^2 \propto k^{-3/2} \]

**ANISOTROPIC wave packets interactions**

**Weak turbulence**: (Galtier et al. 2000)

Resonant interactions dictate \[ k_\perp \gg k_\parallel \]

Pseudo-Alfven waves decouple

Energy cascades in direction of large \[ k_\parallel \]

\[ E(k_\perp) \propto k_\perp^{-2} \]

\[ v_\lambda \propto b_\lambda \propto \lambda^{1/4} \]

\[ v_\lambda \propto b_\lambda \propto \lambda^{1/2} \]
Inertial scalings (cont)

**ANISOTROPIC wave packets interactions**

**Strong turbulence:** (Goldreich & Sridhar 1995; Cho & Vishniac 2000; Boldyrev 2005)

Achieved when **CRITICAL BALANCE**

\[ k_{||} v_A \sim k_{\perp} b_{\lambda} \]

- Fluctuating-fluctuating interactions comparable to mean-fluctuating
- Field lines strongly bent now
- Wave packets are distorted in one interaction not many
- Small wave packets are not guided by the original guide field but the locally distorted field (distorted by larger packets)
- Scale-dependent anisotropy

Spectrum:

\[ E_{GS}(k_{\perp}) \propto k_{\perp}^{-5/3} \]
Inertial scalings (cont)

**ANISOTROPIC wave packets interactions**

**Strong turbulence:** (cont)

BUT numerics find \[ E(k_{\perp}) \propto k_{\perp}^{-3/2} \] Flatter!

**Latest theories explain this:** (Boldyrev 2005, 2006)

Explains longer energy transfer times at small scales by **DYNAMICAL ALIGNMENT**.

\( v \) and \( b \) aligned with scale dependent angle; reduces nonlinear interactions

(Balanced … unbalanced …)
Conclusions

MHD guide-field turbulence very different from hydro

Can be **WEAK** or **STRONG**:
- Weak: mean-fluctuating interactions dominate
- Strong: fluctuating-fluctuating are comparable
- Transition in the inertial cascade? If strength of fluctuating-fluctuating interactions increase with smaller scales, should be a transition from weak to strong turbulence in a sufficiently extended inertial regime (very difficult to simulate).

Energy cascade **ANISOTROPIC**: transfer mostly transverse to the local mean field.

Polarisation vectors tend to align in the transverse plane: consequences for energy spectrum.

Elongation and alignment increase with decreasing scale