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Hydrodynamic Instability

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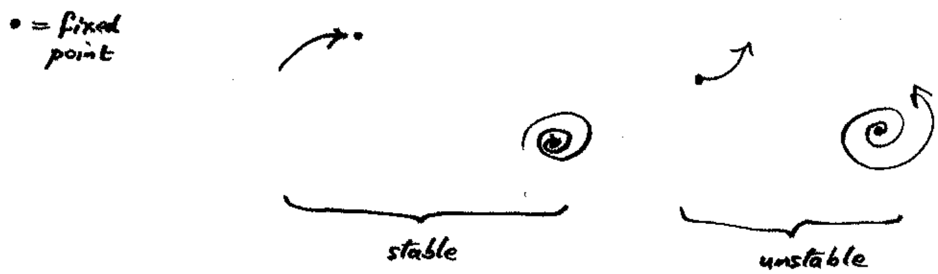
Basic configuration: could be a static equilibrium, a steady state of motion, or an evolving state, satisfying $\mathcal{E}(u) = 0$ $\rightarrow$ solution $u(\underset{\sim}{x}, t;\ u_i(\underset{\sim}{x}, t_0), t_0)$

from some initial conditions $u_i(\underset{\sim}{x}, t_0)$ at $t = t_0$

Let $u_i \rightarrow u_i + \delta u_i$ Then $u \rightarrow u + \delta u$

System is stable if $\left|\frac{\delta u}{\delta u_i}\right|$ bdd for all δu_i as $t \rightarrow \infty$.

I shall consider only stability of steady states: $\frac{\partial}{\partial t} = 0 \rightarrow \mathcal{E}_0$ $\mathcal{E}_0(u_0) = 0$



Linear theory: analysis in the nbd of fixed point. $u \rightarrow u_0 + \delta u$

Linearize in δu.

$$\mathcal{E}(u_0 + \delta u) = \underbrace{\mathcal{E}_0(u_0)}_{=0} + \left.\frac{\delta \mathcal{E}}{\delta u}\right|_{u_0} \delta u + \underbrace{\ldots}_{\text{ignore}} = 0$$

Because u_0 is indep of t (and $\frac{\delta \mathcal{E}}{\delta u}$ not explicitly dependent on t, although it contains time derivatives) can seek solutions of form $e^{\eta t}$ η = growth rate

[There can be systems that evolve otherwise, but I'll not discuss them.]

The $\mathcal{L}\delta u = 0$ $\mathcal{L}$ = Linear differential operator $\left(= \left.\frac{\delta \mathcal{E}}{\delta u}\right|_{u_0}\right)$

Typically a boundary-value problem.

I shall briefly discuss different methods of studying linear stability, illustrating them with some common examples.

<u>Convective instability</u>

The simplest of all problems. Before analysing carefully one should determine the answer — in order to determine whether one has done the calculation correctly, and/or to hone one's intuition.

Push a parcel of fluid downwards distance dz, slowly (to maintain pressure balance), assuming the motion to be adiabatic

Then density of parcel is $\rho_0 + \left(\frac{\partial \rho}{\partial p}\right)_{ad} dp$ ~~z measured downwards~~

Density in ambient fluid is $\rho_0 + \frac{d\rho}{dp} dp$ $dp > 0$ for downwards displacement

System is unstable if density of parcel exceeds that of surroundings

i.e. if $\left(\frac{\partial \rho}{\partial p}\right)_{ad} > \frac{d\rho}{dp}$ or $\left(\frac{\partial \ln\rho}{\partial \ln p}\right)_{ad} > \frac{d\ln\rho}{d\ln p}$

since $\rho, p > 0$

It is usual to define $\gamma = \gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_{ad}$. ∴ define stratification of ambient fluid by $\Gamma = \frac{d\ln p}{d\ln \rho}$

Then $\frac{1}{\gamma} - \frac{1}{\Gamma} > 0$ for instability. — (1) Convective instability criterion

For a <u>liquid</u>, $\gamma = \infty$ and $\frac{1}{\Gamma} < 0$ for instability.

i.e. density increases upwards. Rayleigh-Taylor condition.

Unfortunately some people rewrite condition as $\frac{\Gamma - \gamma}{\Gamma\gamma} > 0$ and infer instability if $\Gamma > \gamma$ or $\Gamma < 0$ and consider them to be two <u>different</u> conditions.

i.e. $\underbrace{\Gamma > \gamma}_{\text{convective instability}}$ is a different instability from $\underbrace{\Gamma < 0}_{\text{Rayleigh-Taylor instability.}}$

Some ~~they~~ even stay that in a convection zone one can have $\Gamma > \gamma$ but not $\Gamma < 0$. I hope that ~~it~~ it is evident that this distinction is spurious.

Many people prefer to write the condition in terms of temperature, because they think of convection mainly in terms of liquids for which density is essentially determined only by T.

Since $\left(\frac{\partial \ln\rho}{\partial \ln p}\right)_{ad} = \left(\frac{\partial \ln\rho}{\partial \ln p}\right)_T + \left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p \left(\frac{\partial \ln T}{\partial \ln p}\right)_{ad}$ if chemical composition is uniform, and $\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_p < 0$

with a similar equation for background state, instability criterion becomes

$$\nabla - \nabla_{ad} > 0 \quad \text{—— (2)} \qquad \text{where} \quad \nabla \doteq \frac{d\ln T}{d\ln p}, \quad \nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_{ad}$$

NB The subscript ad denotes a derivative at const. specific entropy s.

(Now the distinction between the full convective instability criterion and the Rayleigh-Taylor criterion is concealed.)

If the chemical composition, characterized by the mean molecular mass μ, varies in the ambient medium there must be added another term to ∇ to account for the density variation; there is no such term to be added to ∇_{ad} because chemical composition is preserved in the displaced parcel. Simplifying to a perfect gas satisfying $p = \frac{\mathcal{R}\rho T}{\mu}$ (it isn't necessary to do that), condition (1) becomes

$$\nabla - \nabla_{ad} > \frac{d\ln\mu}{d\ln p} \qquad \text{——— (3)}$$

Note that conditions (1 and (3) are identical (for a perfect gas), and it can be shown that they are correct — I'll improve my argument in a moment.

Condition (3) is called the Ledoux criterion by astronomers (because Ledoux was the first astronomer to make the transformation from condition (1).)

Condition (2) is called the Schwarzschild criterion by astronomers, because K. Schwarzschild transferred it from the meteorological literature to the astronomical literature. It is valid only for a chemically homogeneous gas.

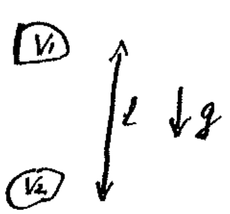
There are conditions that sometimes arise in stars where a truly hydrostatic state is convectively unstable. The normal assumption is that the <u>nonlinear</u> development of the instability moves the background state <u>towards</u> a neutrally stable state — this is common to many, but not all, instabilities. Convective mixing homogenizes the fluid, and the system moves towards conditions (1), (2) and (3) with equality.

There are conditions that sometimes occur at the edges of regions of nuclear burning in stars where complete homogenization is impossible — these regions undergo a form of doubly diffusive ('thermohaline') convection. Astronomers (astrophysicists) argue whether it is towards Schwarzschild neutrality or Ledoux neutrality that the system tends to evolve. The motion is too weak to completely homogenize the fluid. Recent numerical simulations suggest that in general it is neither; the system is much more complicated.

<u>Energy principles</u>

A somewhat better approach is via an energy principle. First I'll be crude.

ρ_0



$\rho_2 = \rho_0 + \frac{d\rho}{dp} dp + \dots$

$dp = -\frac{dp}{dz} l + \dots$

Interchange the contents of two very small volumes V_1 and V_2, expanding or compressing each fluid parcel adiabatically to fully occupy its new volume and calculate the change of energy (potential + internal) ΔE. The volumes are separated by a small vertical distance l. g is the acceleration due to gravity. Each fluid element starts out with the same properties as the ambient medium.

Now minimize ΔE amongst all ratios V_1/V_2. (It is found that the minimizing value leaves the displaced fluid parcels in pressure equilibrium with their surroundings). If $\Delta E_{min} > 0$, it would require external work for the displacement to be affected so the fluid could not do it spontaneously. Therefore (it seems) the system must be stable. The condition turns out to be criterion (1) — with the sign reversed, for stability — in the limit $\ell \to 0$.

It does <u>not</u> follow that the system is unstable if condition (1) is satisfied, because that simple interchange cannot satisfy the equations of fluid dynamics — whether an energetically equivalent disturbance that does satisfy the equations exists requires more careful treatment.

Moreover, it does <u>not</u> follow from this simple analysis that if condition (1) is not satisfied the system is stable, because for stability $\Delta E > 0$ for all possible disturbances, and here I have considered just one class (which don't even satisfy the equations.

The analysis can be tightened by imagining displacements $\underset{\sim}{\xi}(\underset{\sim}{r})$ of the fluid that are continuous and potentially realizable. One could even demand that the linearized equations of motion are satisfied in helping to construct the formula, namely

$$\rho \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} = -\nabla p' - \rho' \nabla \Phi - \rho \nabla \Phi' \qquad (4)$$

$$\delta\rho + \rho \, \mathrm{div}\, \underset{\sim}{\xi} = \rho' + \mathrm{div}(\rho \underset{\sim}{\xi}) = 0 \qquad (5)$$

$$\nabla^2 \Phi' = 4\pi G \rho' \qquad (6)$$

$$\delta p = c^2 \delta\rho \qquad (7)$$

where $\delta\rho$ and ρ' etc are respectively Lagrangian and Eulerian perturbations: $\delta\rho = \rho' + \underset{\sim}{\xi}\cdot\nabla\rho$, and Φ is the gravitational potential. One can either

set $\frac{\partial}{\partial t} = \eta$, multiply the momentum equation by $\underset{\sim}{\xi}^*$ (the complex conjugate of $\underset{\sim}{\xi}$)
i.e. take the scalar product

and use the other equations appropriately, <u>or</u>, construct the energy change directly, to obtain (if Φ' is neglected)

$$\eta^2 = \frac{K}{I} \qquad \text{where } I = \tfrac{1}{2}\int \rho\, \underset{\sim}{\xi}.\underset{\sim}{\xi}^* dV$$ (the integral being over the volume of the region)

$$\text{and} \qquad K = -\Delta E = \tfrac{1}{2}\int \left(\gamma p \chi^2 + 2g\rho \xi \chi + g\rho \frac{d\ln p}{dz} \xi^2\right) dV \qquad (8)$$

where $\chi = \operatorname{div} \underset{\sim}{\xi}$ and $\xi = \underset{\sim}{\xi}.\underset{\sim}{n}$ where $\underset{\sim}{n}$ is a unit vertical vector

The integrand is a quadratic form in ξ, χ which is negative everywhere if

$-\gamma g p \rho \frac{d\ln p}{dz} - g^2\rho^2 < 0.$ i.e. if $\frac{1}{\gamma} - \frac{1}{\Gamma} > 0$ for stability.

This is a genuine condition for stability, because $\Delta E > 0$ for <u>all possible</u> $\underset{\sim}{\xi}$, which must include the solutions to the equations. However, the converse is not necessarily the condition for instability because it could be that none of the functions $\underset{\sim}{\xi}$ that cause $\Delta E < 0$ are realizable (i.e. satisfy the equations of motion). It can be shown by analysing the equations of motion more carefully that condition (1) is both necessary and sufficient.

It was shown by Lebovitz that condition (1) is necessary and sufficient even if the perturbation Φ' to the gravitational potential is not ignored. It is easy to convince oneself that that must be the case in the vicinity of states with $\Gamma = \gamma$ because the neutral disturbance (i.e. the most unstable disturbance when $\Gamma = \gamma$) has no buoyancy, no density perturbation, and hence $\Phi' = 0$. One might imagine a situation far from $\Gamma = \gamma$ that might be gravitationally unstable, but that would not be a stable star.

Jeans' Instability

The simplest analysis of gravitational instability was studied by Jeans, who considered linearized perturbations to a large, almost uniform gas cloud. The linearized momentum equation is

$$\rho \frac{\partial \underset{\sim}{u}}{\partial t} = \frac{\partial}{\partial t}(\rho \underset{\sim}{u}) = -\nabla p' - \rho \nabla \Phi' - \rho' \nabla \Phi \qquad (9)$$

in which ρ and Φ are equilibrium (time independent) quantities. Jeans considered perturbations with a spatial scale k^{-1} much smaller than the scale H of variation of the cloud, which permitted the last term on the rhs to be neglected. Then, taking the divergence of the momentum equation and using

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}(\rho \underset{\sim}{u}) = 0 \quad \text{and} \quad \nabla^2 \Phi' = 4\pi G \rho' \quad \text{and} \quad p' = c^2 \rho'$$

to eliminate p', $\underset{\sim}{u}$ and Φ' in favour of ρ', where $c^2 \left(= \frac{\gamma p}{\rho}\right)$ is the square of the sound speed, yields

$$\frac{\partial^2 \rho'}{\partial t^2} = \nabla^2 (c^2 \rho') + 4\pi G \rho \rho' \qquad (10)$$

which, if $\rho' \propto e^{i\underset{\sim}{k}\cdot\underset{\sim}{x} - i\omega t}$ yields the dispersion relation

$$\omega^2 = c^2 k^2 - 4\pi G \rho \qquad (11)$$

whence modes with $k < \frac{\sqrt{4\pi G \rho}}{c} \equiv \frac{2\pi}{\lambda_J}$ (but with $k^{-1} > H$) are unstable.

Many have criticized Jeans for neglecting $\nabla \Phi$, stating that $\nabla^2 \Phi = 4\pi G \rho$ and cannot be zero, and have called the approximation the Jeans swindle. This is unfair because, as I have pointed out, Jeans did recognize this and argued for asymptotically large clouds. Ledoux considered a plane parallel slab in strict gravitational (hydrostatic balance) and showed that the system is unstable for $k < \frac{2\pi}{\tilde{\lambda}_J}$, where $\tilde{\lambda}_J$ is a characteristic value of λ_J. The instability can also be derived (rigorously) in an expanding universe. In recent times asymptotic justification for the criterion has been made more formal.

Kelvin's circulation theorem (generalized), in differential form

In most astrophysical circumstances viscous stresses are negligible and the motion is well approximated by being either adiabatic or isothermal. In either case one can write

$$\frac{D\ln p}{Dt} = c^2 \frac{D\ln\rho}{Dt} \qquad \text{where} \quad c^2 = \gamma\frac{p}{\rho} \quad \text{and either } \gamma = \gamma_1 = \left(\frac{\partial \ln p}{\partial \ln\rho}\right)_s$$

$$\text{or } \gamma = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_T$$

where s is specific entropy and T is temperature. For a perfect monatomic unionizing gas, $\gamma_1 = \frac{5}{3}$ and $\left(\frac{\partial \ln p}{\partial \ln\rho}\right)_T = 1$.

A useful consequence of adiabatic motion is an equation governing the evolution of vorticity $\underset{\sim}{\omega} = \operatorname{curl}\underset{\sim}{u}$. Taking the curl of the momentum equation in the form

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \underset{\sim}{u}\wedge\underset{\sim}{\omega} = -\nabla(h + \tfrac{1}{2}u^2 + \Phi) + T\nabla s \qquad \text{——— (12)}$$

under a conservative force with potential Φ, where h is specific enthalpy $\epsilon + p/\rho$ gives

$$\frac{\partial \underset{\sim}{\omega}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{\omega} + \underset{\sim}{\omega}\,\operatorname{div}\underset{\sim}{u} = \underset{\sim}{\omega}\cdot\nabla\underset{\sim}{u} + \nabla T\wedge\nabla s . \qquad \text{——— (13)}$$

Multiplying this by ρ and subtracting $\underset{\sim}{\omega}$ times the continuity equation $\left(\frac{D\rho}{Dt} + \rho\operatorname{div}\underset{\sim}{u} = 0\right)$ eliminates $\operatorname{div}\underset{\sim}{u}$ leaving

$$\frac{D}{Dt}\left(\frac{\underset{\sim}{\omega}}{\rho}\right) = \frac{\underset{\sim}{\omega}}{\rho}\cdot\nabla\underset{\sim}{u} + \rho^{-1}\nabla T\wedge\nabla s . \qquad \text{——— (14)}$$

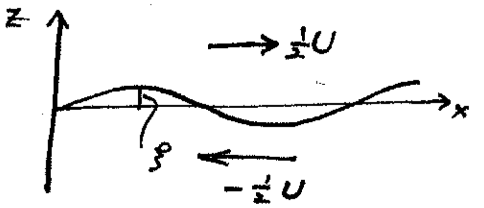
It is evident that in a homentropic fluid ($\nabla s = \underset{\sim}{0}$), $\frac{\underset{\sim}{\omega}}{\rho}$, which is sometimes called the potential vorticity, moves with the fluid, for then it satisfies the same equation as an advected line element. But if $\nabla s \neq \underset{\sim}{0}$, vorticity can be generated (by buoyancy) and the vorticity is less constrained. However, the equation for $\frac{\underset{\sim}{\omega}}{\rho}$ can be combined with the adiabatic constraint by multiplying it by ∇s (scalar product) and adding $\frac{\underset{\sim}{\omega}}{\rho}\cdot\frac{D\nabla s}{Dt}$ yielding, with the help of the adiabatic constraint $\frac{Ds}{Dt} = 0$,

$$\frac{D}{Dt}\left(\frac{\underset{\sim}{\omega}\cdot\nabla s}{\rho}\right) = 0. \qquad \text{——— (15)}$$

One sees as a consequence that in layerwise 2-d (horizontal) flow in e.g. a star in which ∇s is vertical and ∇s and ρ hardly vary horizontally, the vertical component of $\underset{\sim}{\omega}$ is conserved following the motion.

Kelvin-Helmholtz instability

Vortex sheets are unstable. Consider the simple case of an initially uniform flow $\frac{1}{2}U$ in the x direction for $z>0$ and $-\frac{1}{2}U$ for $z<0$ of a uniform incompressible inviscid fluid. Because $\underset{\sim}{\omega} = \underset{\sim}{0}$ initially, except at $z=0$, $\underset{\sim}{\omega}$ remains zero, and one can set $\underset{\sim}{u} = \nabla\phi$. $\operatorname{div}\underset{\sim}{u} = 0 \Rightarrow \nabla^2\phi = 0$ and setting $\phi = \phi_0 e^{ikx+\eta t}$ (η is the growth rate) — it is straightforward to show that a perturbation of the form $e^{ikx+ily+\eta t}$ with $l \neq 0$ is less unstable,* because kinetic energy pumped into y flow is wasted — yields $\phi_0(z) = Ae^{\mp kz}$ for $z \gtrless 0$. (* This is called Squire's Theorem)

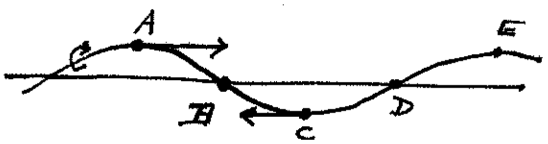


Continuity of p across the vortex sheet implies continuity of ϕ, and the constraint that $\frac{D\xi}{Dt} = \frac{\partial\phi}{\partial z}$ then yields $(\eta \pm \frac{1}{2}Uik)\xi = \mp k\phi\big|_{z=0}$, the upper and lower signs referring to the upper and lower ~~layers~~ regions of fluid. Eliminating ξ and $\phi \Rightarrow$

$$\eta = \pm\tfrac{1}{2}kU$$

one of which is positive. $\therefore$ the sheet is unstable.

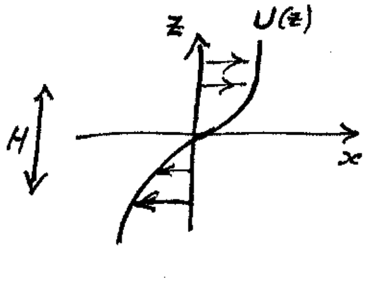
Batchelor explains the instability as follows:



Vorticity ↻ is advected to right from A and to left from C to augment circulation near B which augments the clockwise turning of the sheet at B and thereby enhances the disturbance.

What about advection from C and E towards D, you may ask. That is the dominant process controlling the stable mode, with $\eta = -\frac{1}{2}kU$.

More general shear flows (incompressible, uniform)



(a)

Continuous rectilinear flows $U(z)$ have received much attention. I shall be brief. Once again, in view of Squire's Theorem, consider perturbations in the x-z plane, but now described by the stream function ψ

$(u, w) = \left(\frac{\partial\psi}{\partial z}, -\frac{\partial\psi}{\partial x}\right)$, with $\psi \propto e^{ik(x-ct)}$

Then $$(U-c)\left(\frac{\partial^2\psi}{\partial z^2} - k^2\psi\right) - \frac{d^2U}{dz^2}\psi = 0 \qquad (16)$$

Dividing by $U-c$, multiplying by ψ^*, the complex conjugate of ψ, and integrating wrt $z \Rightarrow$

$$\int \left(|\psi'|^2 + k^2|\psi|^2\right)dz + \int \frac{U''}{U-c}|\psi|^2 dz = 0 \qquad (17)$$

where the prime denotes differentiation wrt z.

Rayleigh considered the imaginary part of this equation: $\quad c_i \int \frac{U''}{|U-c|^2}|\psi|^2 dz = 0$ — (18)

where $c_i = Im(c)$, $\Rightarrow$ if $c_i \neq 0$, U'' must change sign somewhere.

Fjørtoft subsequently considered the real part, from which, with the help of (18) one can obtain

$$\int \frac{U''(U-U_s)}{|U-c|^2}|\psi|^2 dz = -\int \left(|\psi'|^2 + k^2|\psi|^2\right)dz < 0$$

where U_s is the value of U where $U''=0$, from which it follows that

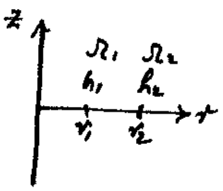
$U''(U-U_s) < 0$ somewhere if the ~~fluid~~ flow is unstable. — (19)

(actually U_s can be any constant if the flow is unstable, in view of (18).)

These are often called the inflexion point criterion, but they are more appropriately considered to be a vorticity criterion: for instability the magnitude of the vorticity must have a local maximum.

It is evident that a flow such as that drawn at the bottom of the previous page (which satisfies condition 19) must be unstable because for small enough k the vertical scale shrinks to zero (i.e. $Hk \to 0$) and the flow looks like a vortex sheet subject to the Kelvin-Helmholt instability. But if the flow is confined between two boundaries, at $z = \pm z_0$, say, then the flow can be stabilized if z_0/H is sufficiently small. Condition (19) is necessary but not sufficient. An explicit analysis has been carried out for the case with $U(z) = \tanh(z/z_0) - \frac{1}{2}$.

Centrifugal instability (again, according to Rayleigh) Invisid, uniform, incompressible



Consider axisymmetric disturbances about a flow with angular velocity $\Omega(r)$ about the z axis. Rayleigh considered the equations of motion, as for the shear instability, but I'll use an energy principle, which is very simple.

Consider two small (thin) rings of fluid about r_1 and r_2 with ang. mom. $h_1 = r_1^2\Omega_1$ and $h_2 = r_2^2\Omega_2$ and energies $h_1^2/2r_1^2$ and $h_2^2/2r_2^2$ per unit mass and calculate the kinetic energy change ΔT after interchanging them conserving their angular momentum.

The system is stable if $\Delta T > 0$, which is so if $\frac{dh^2}{dr} < 0$ everywhere. —— (20)

On Easter day, 1965, I found myself in the library of the Department of Applied Mathematics and Theoretical Physics in Cambridge, sheltering from the snow and without a partner, because my wife was in hospital (working as a nurse). I was trying to reconcile the two instability criteria of Rayleigh, which I have just described, and realized that if condition (19) was written in terms of vorticity, rather than the more common point of inflexion, it was true also of nonaxisymmetric disturbances of the rotating system for which I discussed centrifugal instability. And then I read an interesting footnote in Rayleigh's paper on the shear instability, pointing out that there is a corresponding criterion for swirling flow (which he didn't write down) but worded in such a way (as I realized only after I could appreciate it by having done the calculation myself) that it was clear that Rayleigh knew what that criterion is. Vorticity hadn't yet been invented, so Rayleigh would not have described his criterion in quite the same terms as I. (Moreover, he did not have Fjørtoft's refinement.) As I was pondering Rayleigh's remark, Donald Lynden-Bell arrived, also sheltering from the snow and partnerless because his wife was in the same hospital as mine (having a baby). Donald was interested to see me reading Rayleigh, and told me of his interest in the role of circulation per unit mass (which he subsequently called circass) which is an integral of $\rho^{-1}\underset{\sim}{\omega}$ and satisfies a generalized Kelvin circulation theorem that is essentially an integral form of equation (14) in the case $\nabla s = \underset{\sim}{0}$. We discussed it in some detail, Donald suggesting that there should be a generalization of Rayleigh's shear-flow criterion, and after Donald had gone home (we were to meet subsequently to investigate an idea about vorticity expulsion by turbulence, which arose from our Easter-Day discussions about turbulent mixing of conserved quantities such as vorticity) I showed that the problem generalized, (when appropriately written in terms of what one might call potential vorticity) to shear flow confined between two boundaries whose separation (with respect to z) varied slowly with x. That also led to equation (15).

Much more recently, Pascale Garaud demonstrated the same criterion for 2-d (horizontal) flow in a differentially rotating star, and went on to demonstrate by weakly nonlinear theory that the angular momentum transport by the instability is in such a direction as to ~~restore~~ move the differential rotation towards neutrality. This is another example of unstable systems seeking their neutrally stable configuration.

Richardson Criterion

Consider a shear flow such as that to which equation (16) pertains but which in addition is under gravity g (in the negative z direction) and in which the fluid density ρ varies with z. One can estimate a criterion by an interchange energy argument in which the total linear momentum is conserved, finding the system to be stable if

$$R_i := -\frac{g \, d\ln\rho/dz}{(dU/dz)^2} > \frac{1}{4}. \qquad (21)$$

More generally one can consider the interchange argument for adiabatic perturbations to a compressible fluid, finding stability for

$$\frac{1}{\Gamma} - \frac{1}{\gamma} > \frac{c^2}{4\gamma g^2}\left(\frac{dU}{dz}\right)^2, \qquad (22)$$

which can be rewritten to make it look more like the Richardson condition (21), rather than the convective stability condition (1), as

$$N^2 > \frac{1}{4}\left(\frac{dU}{dz}\right)^2 \qquad (23)$$

for stability, where N is the buoyancy frequency (Brunt-Väisälä frequency) defined by

$$N^2 = g\left(\frac{1}{H} - \frac{g}{c^2}\right) \qquad \text{where} \quad H^{-1} = -\frac{d\ln\rho}{dz}. \qquad (24)$$

The instability associated with the vortex sheet requires sufficient convectively stable stratification to quench it.

Stabilization of convection by a magnetic field

The addition of a magnetic field to the convective stability problem renders the problem much more difficult because in the absence of the field disturbances can be localized and lead to a local criterion, whereas a magnetic connects different regions, causing the instability to be generically global. However, it is possible to obtain locally applicable criteria from a energy principle by jettisoning selected positive semidefinite terms and demanding that the residual component of the energy increment be positive. For a fluid with zero magnetic diffusivity, equation (8) becomes

$$\Delta E = \frac{1}{2}\int \left(Q^2 - \underset{\sim}{j} \cdot \underset{\sim}{Q} \wedge \underset{\sim}{\xi} + \gamma p \chi^2 - 2 g \rho \xi \chi - g\rho \frac{d\ln\rho}{dz} \xi^2 \right) dV \qquad \text{———} \quad (25)$$

where $\qquad Q = \mathrm{curl}\,(\underset{\sim}{\xi} \wedge \underset{\sim}{B})$ and $\underset{\sim}{j} = \mathrm{curl}\,\underset{\sim}{B}$.

This equation was analysed by Tayler and one of his students (officially his first) for the case of a vertical field that depended on one horizontal coordinate x or a uniform inclined field (it can be analysed similarly for an axisymmetric vertical field with vertical axis of symmetry, which may be considered to represent a starspot), obtaining a criterion of the form

$$S^{-1}\Delta E > \frac{1}{2}\int \left[F\left(\frac{\partial \xi}{\partial z}\right)^2 - G\xi^2 \right] dz \qquad \text{(per unit surface area } S\text{)}$$

where $F \geq 0$ and G is a complicated function of the stratification. Evidently, if $G < 0$ the system is stable, but that criterion is difficult to interpret. The situation can be alleviated by adding an arbitrary function to the integrand whose integral is identically zero, and by judicious choice of that function a more palatable sufficient condition for stability can be derived. For example, for a uniform inclined field or a vertical horizontally vary field the stratification is stable to convection if

$$\frac{1}{\gamma} - \frac{1}{\Gamma} < \frac{B_v^2}{\gamma(B_v^2 + p)} \qquad \text{————} \quad (26)$$

i.e. $$\frac{p}{B_v^2 + p}\,\frac{1}{\gamma} - \frac{1}{\Gamma} < 0\,, \qquad \text{————} \quad (27)$$

where B_v is the vertical component of the magnetic field $\underset{\sim}{B}$. Condition (26) shows how the field adds to the energy required to effect the disturbance, by weakening (according to 27) the ability of the fluid to expand (compress) as it rises (sinks).

Criteria (26) and (27) are applicable locally, but, like (the inverse of) condition (1) they are not truly local (as derived here) because they must be satisfied everywhere to ensure stability. (Actually, condition (1) is truly local, because, as I pointed out earlier, it has been shown that the system is unstable if condition (1) is satisfied anywhere.) However it is tempting to believe that convection is actually suppressed in a spot in which conditions (26) and (27), or some alternative sufficient condition, are satisfied, whilst convection can proceed outside. Certainly sunspots provide evidence that that is so.