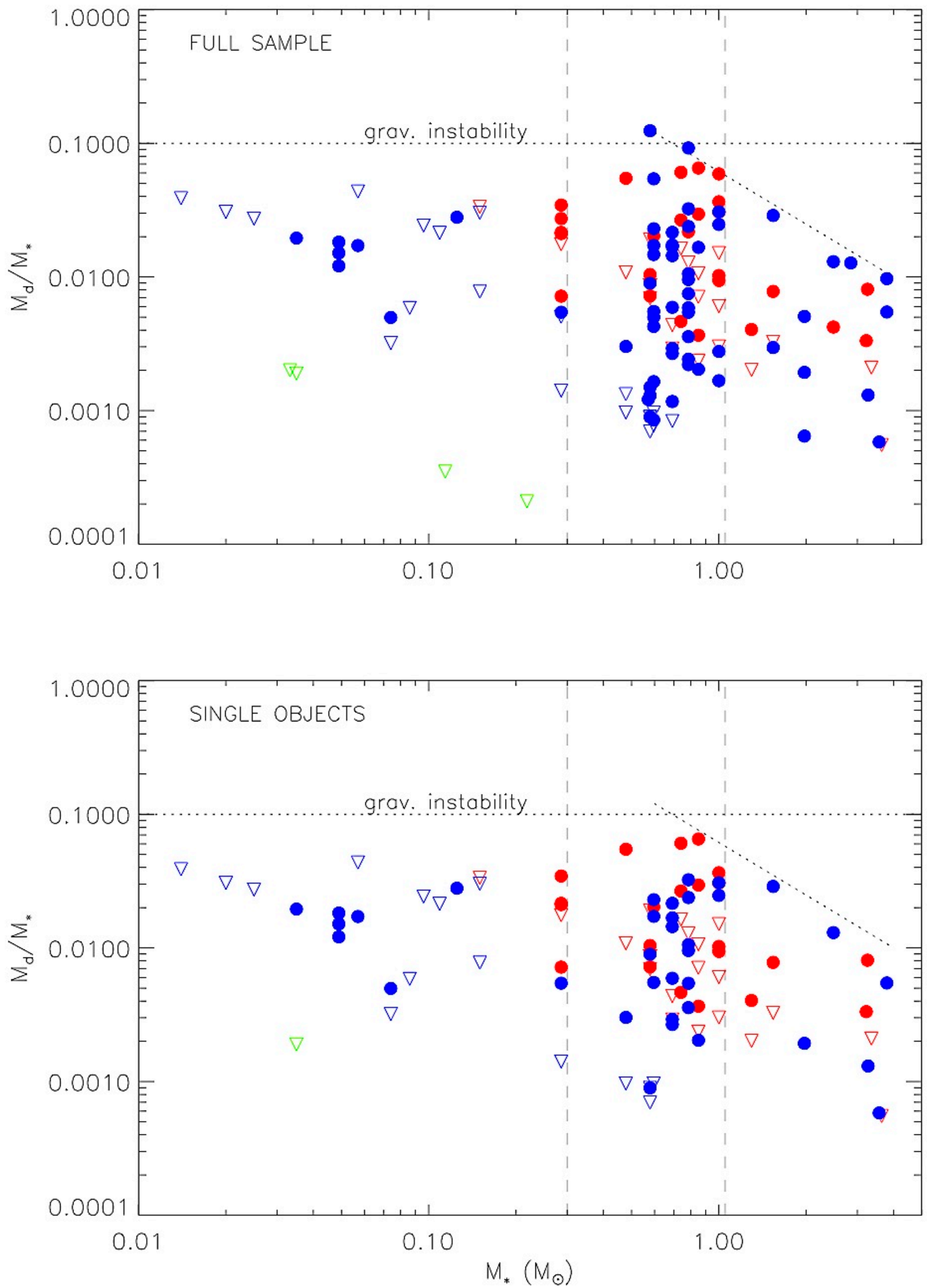
POPULATION INFERENCE WITH UPPER LIMITS:

BAYESIAN ANALYSIS

as applied to

DISK MASSES in VERY LOW MASS STARS & BROWN DWARFS

Subhanjoy Mohanty (Imperial College London)



$M_{dust} = F_\lambda D^2 / [\kappa_\lambda B_\lambda (T_{dust})]$

assume initially that:

$\kappa_\lambda \propto \lambda^{-1}$ with $\kappa_{1.3\ mm} = 2\ cm^2\ g^{-1}$

(Beckwith et al. 1990)

$T_{dust} = 20$ K

(Andrews & Williams 2005, 2007)

gas:dust ratio = 100:1 (ISM)

BAYESIAN ANALYSIS

Consider a specific model $\mathcal{M}_i$ invoked to explain a set of data $\mathcal{D}$.

PRIOR probability we assign to the veracity of the particular model (before comparing to the data)
(encapsulates all our pre-existing knowledge / biases)

LIKELIHOOD of the obtained data, given the model

$$P(\mathcal{M}_i|\mathcal{D}) = \frac{P(\mathcal{M}_i)\, P(\mathcal{D}|\mathcal{M}_i)}{P(\mathcal{D})}$$

POSTERIOR probability of the particular model, given the data
(this is the quantity we want)

EVIDENCE $P(\mathcal{D}) \equiv \sum_i P(\mathcal{M}_i)\, P(\mathcal{D}|\mathcal{M}_i)$

(normalization factor to ensure $\sum_i P(\mathcal{M}_i|\mathcal{D}) = 1$ **)**

OK, now specifically consider:

A model $\mathcal{M}_i$ completely described by N parameters $[\theta_N]_i$.
Data with d measured values $[\hat{m}_d]$ and errors $[\sigma_d]$, and u upper limits $[\hat{m}_{lim,u}]$ and errors $[\sigma_u]$.

1) **Ignore "evidence": cancels for comparing *relative* probabilities within 1 model**
2) **Assume a flat prior (same for all *i*): no prior preference for any *i*.**

Then the posterior is simply proportional to the total likelihood:

$$P([\theta_N]_i \,|\, [\hat{m}_d], [\hat{m}_{lim,u}]) \propto P([\hat{m}_d], [\hat{m}_{lim,u}] \,|\, [\theta_N]_i)$$

where the total likelihood on the RHS is simply the product of the likelihood of each measured value or upper limit:

$$P([\hat{m}_d], [\hat{m}_{lim,u}] \,|\, [\theta_N]_i) = \left[\prod_{d=1}^{n_d} P(\hat{m}_d \,|\, [\theta_N]_i)\right] \times \left[\prod_{u=1}^{n_u} P(\hat{m}_{lim,u} \,|\, [\theta_N]_i)\right]$$

The only subtlety in evaluating this is that the true value of each data point must be taken into account (instead of measured value or upper limit), since we want the underlying distribution; we do so by considering the known noise, as follows.

MEASURED VALUES

Assume that:

1) True value m_d is non-negative
2) Noise is additive and Gaussian

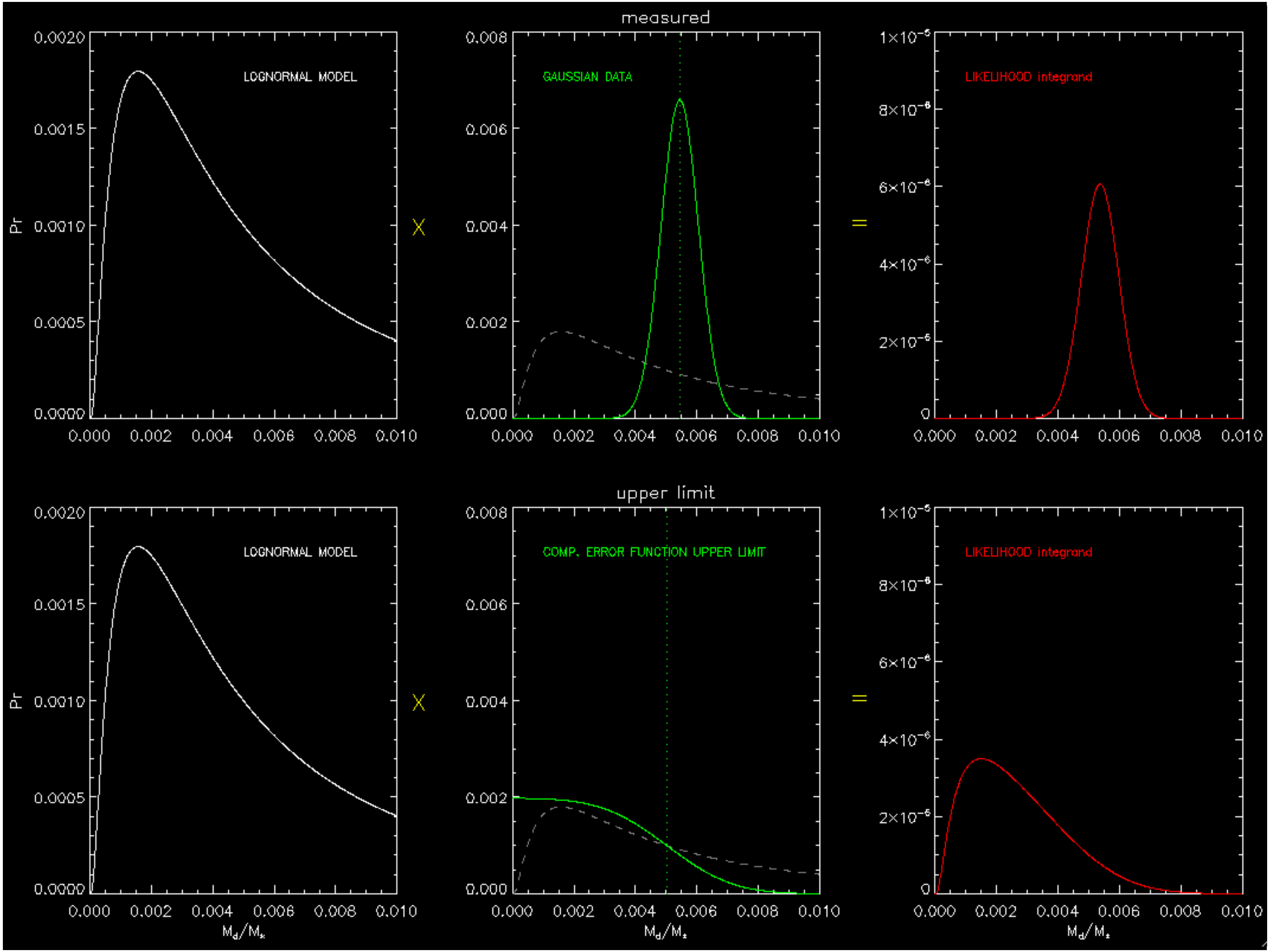
Then for each measured value, the individual likelihood is:

$$P(\hat{m}_d \,|\, [\theta_N]_i) = \int_0^\infty P(m_d \,|\, [\theta_N]_i) \, \frac{1}{(2\pi)^{1/2}\sigma_d} \exp\left[-\frac{1}{2}\left(\frac{\hat{m}_d - m_d}{\sigma_d}\right)^2\right] \mathrm{d}m_d$$

Must usually be evaluated numerically (after model is specified), but straightforward since integral only significant over range:

$\mathrm{MAX}(0, \hat{m}_d - 3\sigma_d) \lesssim m_d \lesssim (\hat{m}_d + 3\sigma_d)$.

NOTE: measured value may well be negative, only the true value is required to be non-negative.



measured
LOGNORMAL MODEL
GAUSSIAN DATA
LIKELIHOOD integrand
Pr
X
=
upper limit
LOGNORMAL MODEL
COMP. ERROR FUNCTION UPPER LIMIT
LIKELIHOOD integrand
Pr
X
=
M_d/M_*
M_d/M_*
M_d/M_*

UPPER LIMITS

Upper limits complicate the likelihood calculation slightly, because an upper limit (by definition) is consistent with any case in which the unreported *measurement* (true value scattered by the noise) is lower than the reported limit. So must marginalize (integrate) over not only the un*known* true value but also the un*reported* measurement.

Again assuming additive and Gaussian noise, the individual likelihood is:

$$P(\hat{m}_{lim,u} \,|\, [\theta_N]_i) = \int_0^{\infty} P(m_u \,|\, [\theta_N]_i) \left\{ \int_{-\infty}^{\hat{m}_{lim,u}} \frac{1}{(2\pi)^{1/2}\sigma_u} \exp\left[-\frac{1}{2}\left(\frac{\hat{m}_u - m_u}{\sigma_u}\right)^2\right] \mathrm{d}\hat{m}_u \right\} \mathrm{d}m_u$$

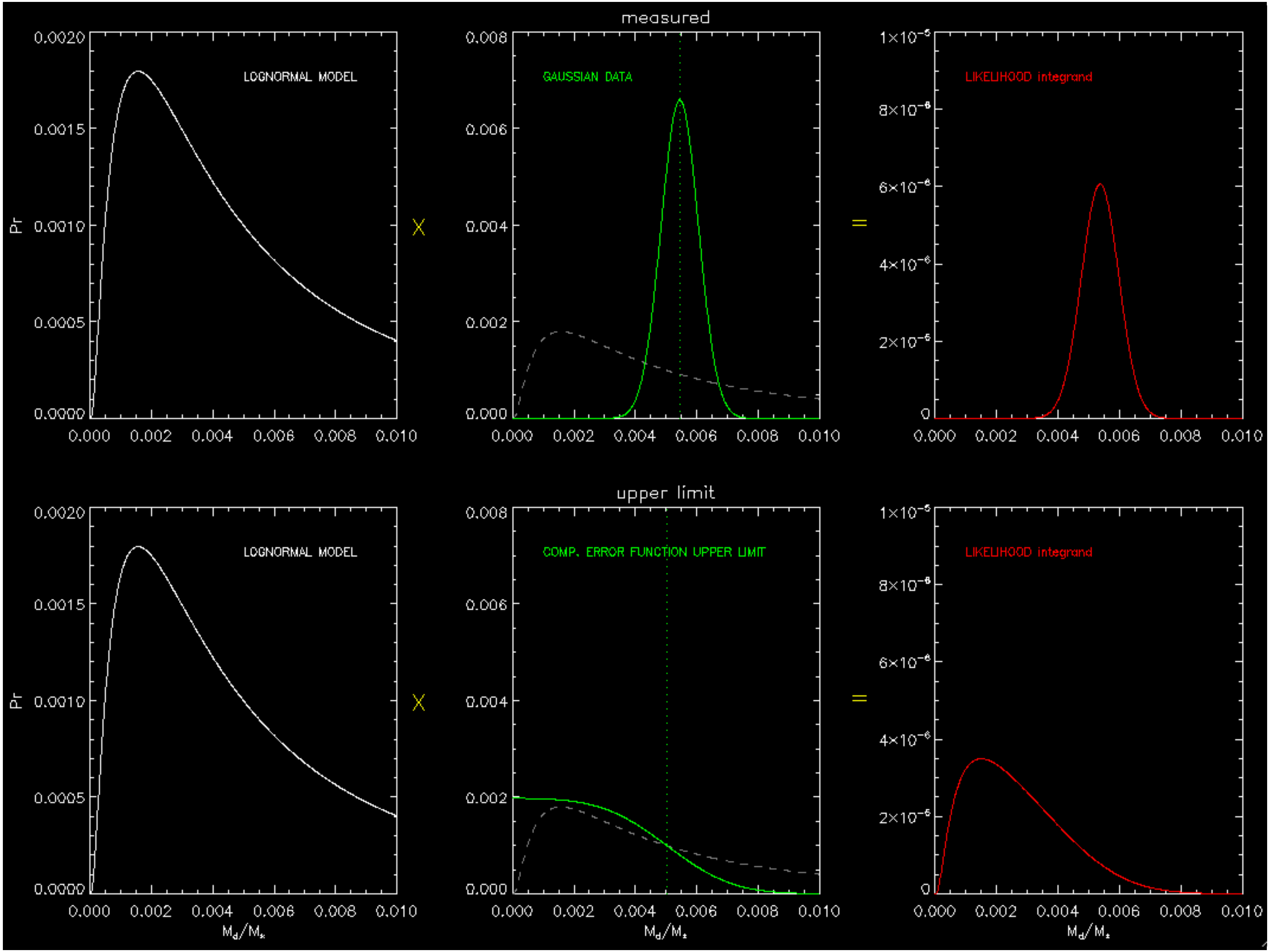
where the inner integral is:

$$\int_{-\infty}^{\hat{m}_{lim,u}} \frac{1}{(2\pi)^{1/2}\sigma_u} \exp\left[-\frac{1}{2}\left(\frac{\hat{m}_u - m_u}{\sigma_u}\right)^2\right] \mathrm{d}\hat{m}_u = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\hat{m}_{lim,u} - m_u}{\sigma_u}\right)$$

Complementary error function! Of course!

so the likelihood simplifies to:

$$P(\hat{m}_{lim,u} \,|\, [\theta_N]_i) = \int_0^{\infty} P(m_u \,|\, [\theta_N]_i)\, \frac{1}{2}\,\mathrm{erfc}\left(\frac{\hat{m}_{lim,u} - m_u}{\sigma_u}\right) \mathrm{d}m_u$$

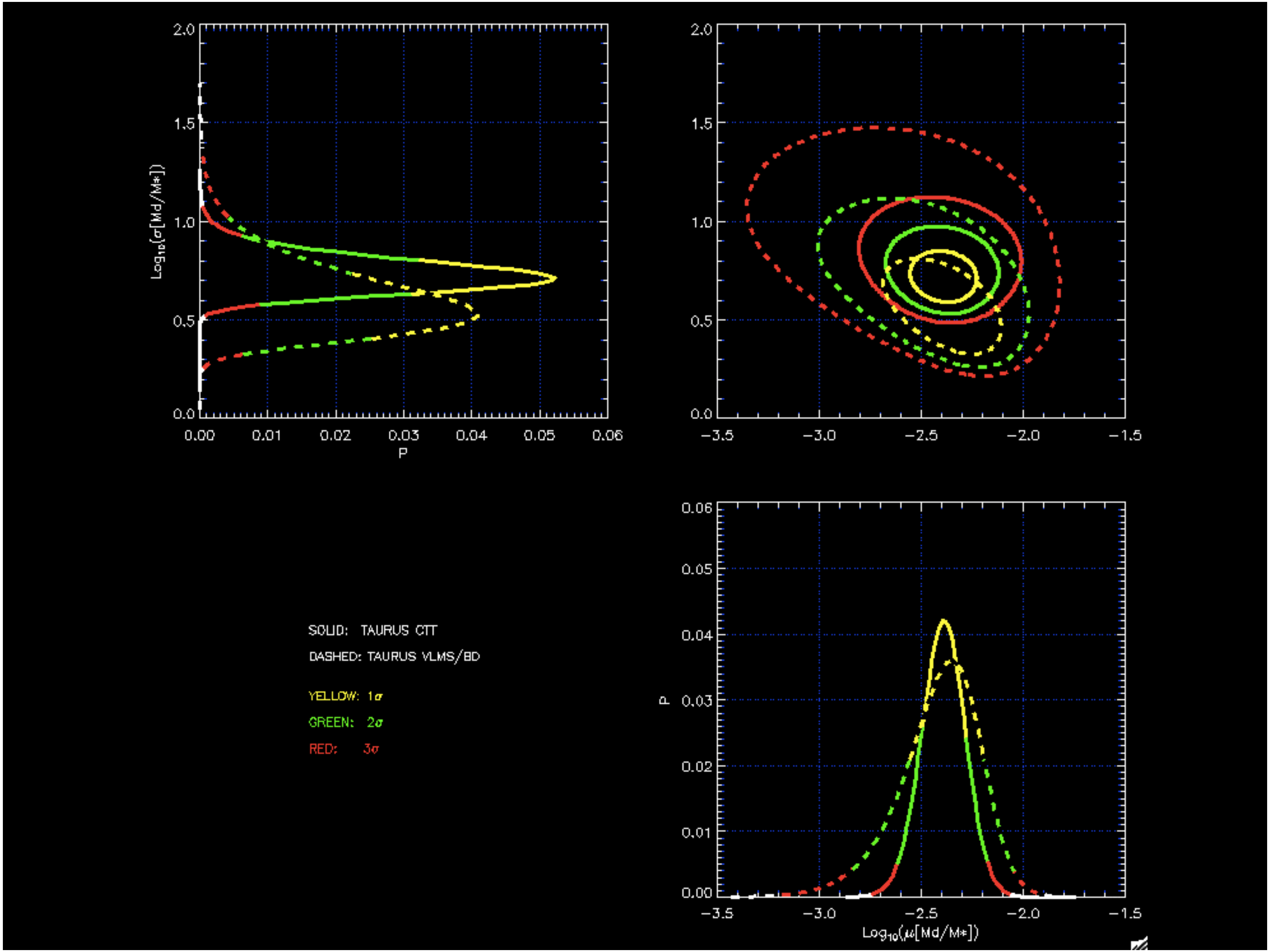


measured
LOGNORMAL MODEL
GAUSSIAN DATA
LIKELIHOOD integrand
upper limit
COMP. ERROR FUNCTION UPPER LIMIT
Pr
X
=
M_d/M_*

A REASONABLE MODEL: LOGNORMAL

We assume that the underlying distribution of M_d/M_* is a lognormal (justified later in §6), specified by 2 parameters: $\theta_{1,i} \equiv \mu_i$ (the mean), and $\theta_{2,i} \equiv s_i$ (the standard deviation). The probability of any particular positive value m ($\equiv M_d/M_*$) in this distribution is:

$$P(m \,|\, [\mu, s]_i) \equiv \frac{1}{(2\pi)^{1/2}\, m\, s_i} \exp\left[-\frac{1}{2}\left(\frac{\ln(m) - \mu_i}{s_i}\right)^2\right]$$

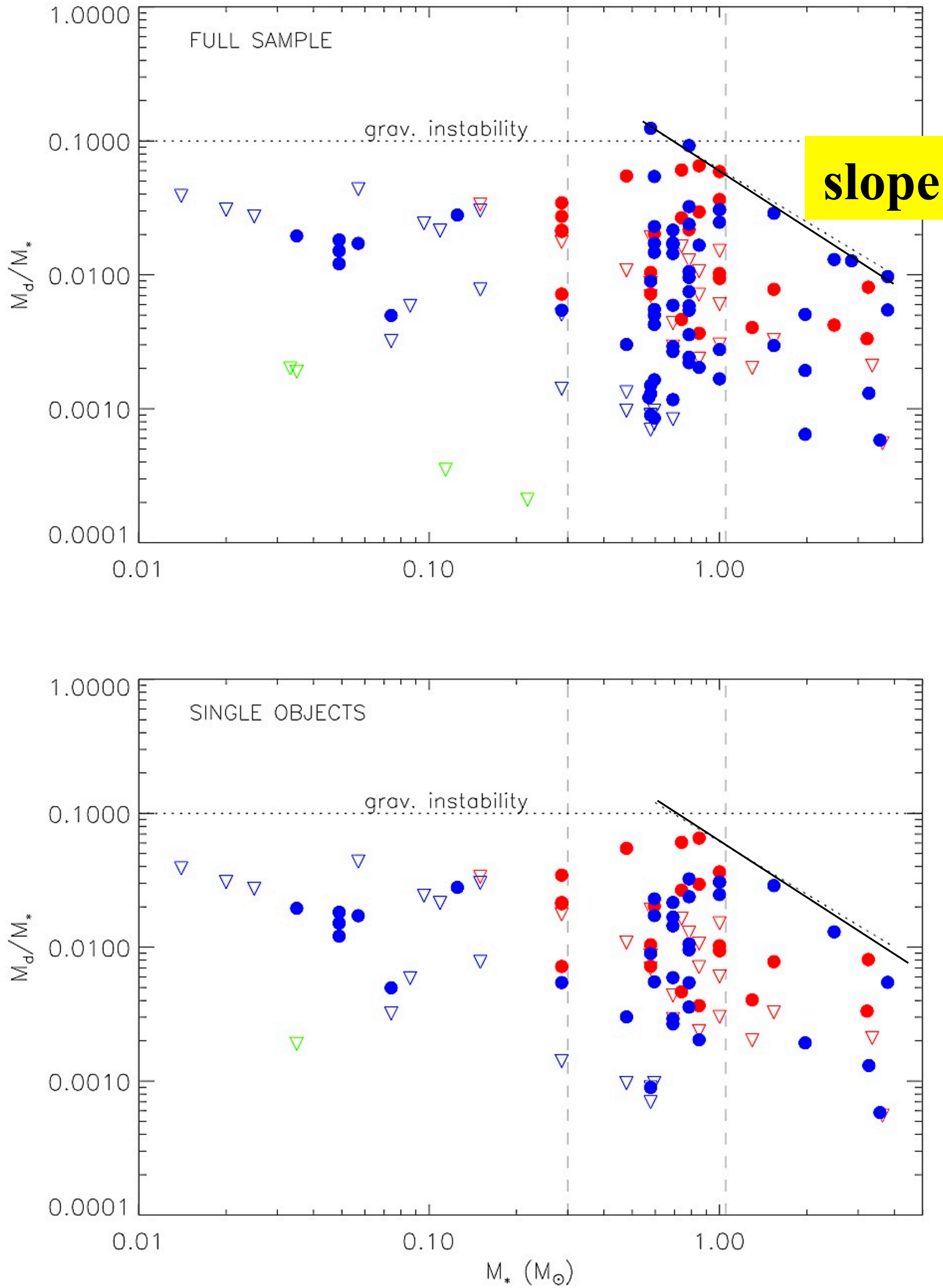


Log10(σ[Md/M*])
P
Log10(μ[Md/M*])
SOLID: TAURUS CTT
DASHED: TAURUS VLMS/BD
YELLOW: 1σ
GREEN: 2σ
RED: 3σ

CONCLUSIONS I

1) Use Bayes for rigorous *and* intuitive stats

2) Please provide actual measurements: the 3σ or 5σ or whatever-σ upper limit cutoff is dogma that throws away honest-to-goodness, perfectly useful data



slope: -1.5

grain growth ?

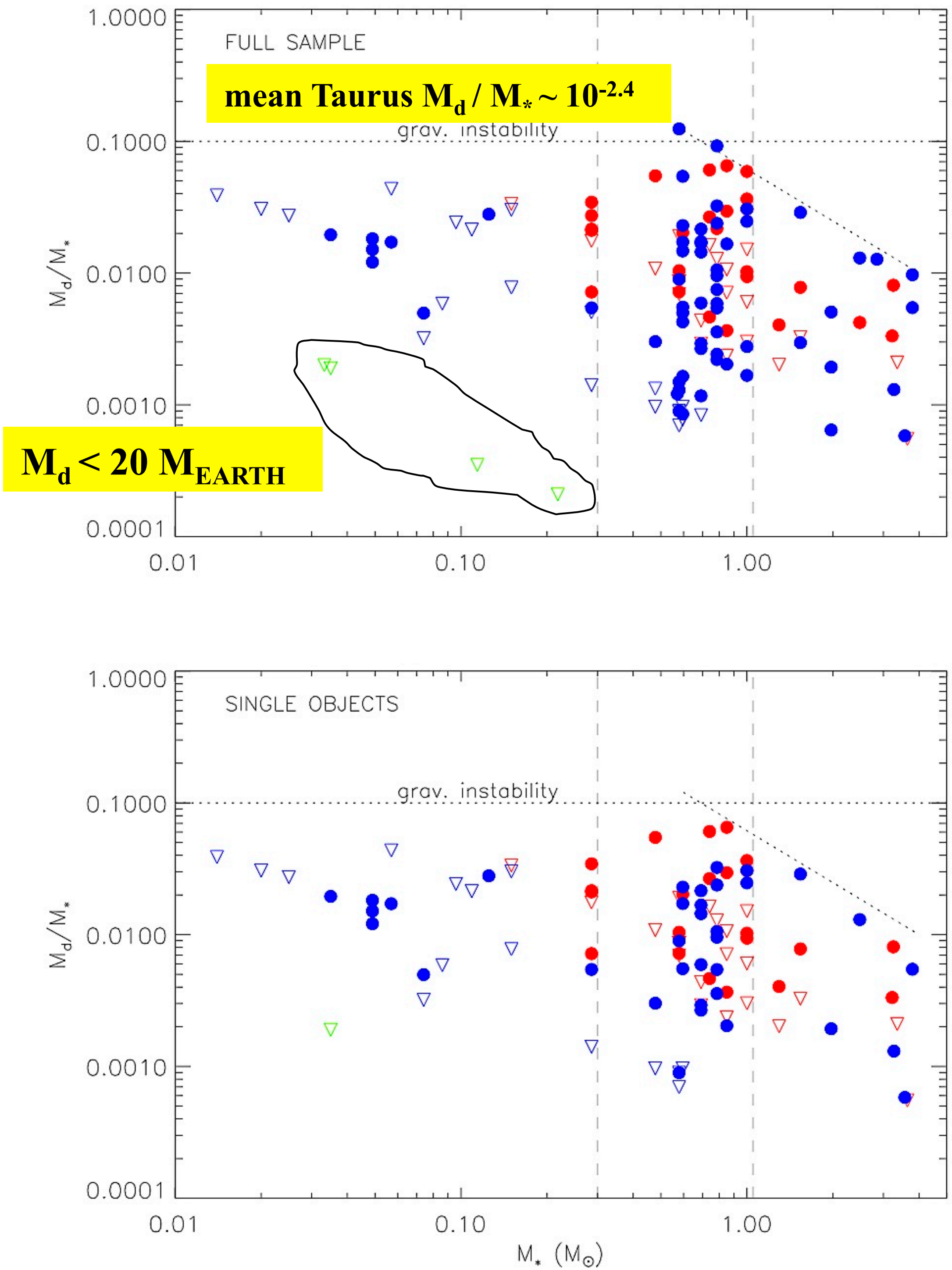
$$M_d(t) = M_d(0) \,/\, [1 + (t \,/\, t_c)],$$
$$t_c \equiv (R_d \,/\, M_*)^{1/2} \,/\, [K\, M_d(0)]$$

(Dominik & Decin 2003
Greaves & Rice 2011)

$\Rightarrow$

for $t >> t_c$ (~0.1 Myr)
and K, R_d ~independent of M_*:

$$M_d(t) \,/\, M_* \propto t^{-1}\, M_*^{-1.5}$$



TWA:

BDs: M_d / M_* ~ mean Taurus / 3

VLMS: M_d / M_* ~ mean Taurus / 10

age: ~ mean Taurus / 10

ACCRETION for a given M_* :

(assuming α radially constant, and $T_d \propto r^{-1/2}$):

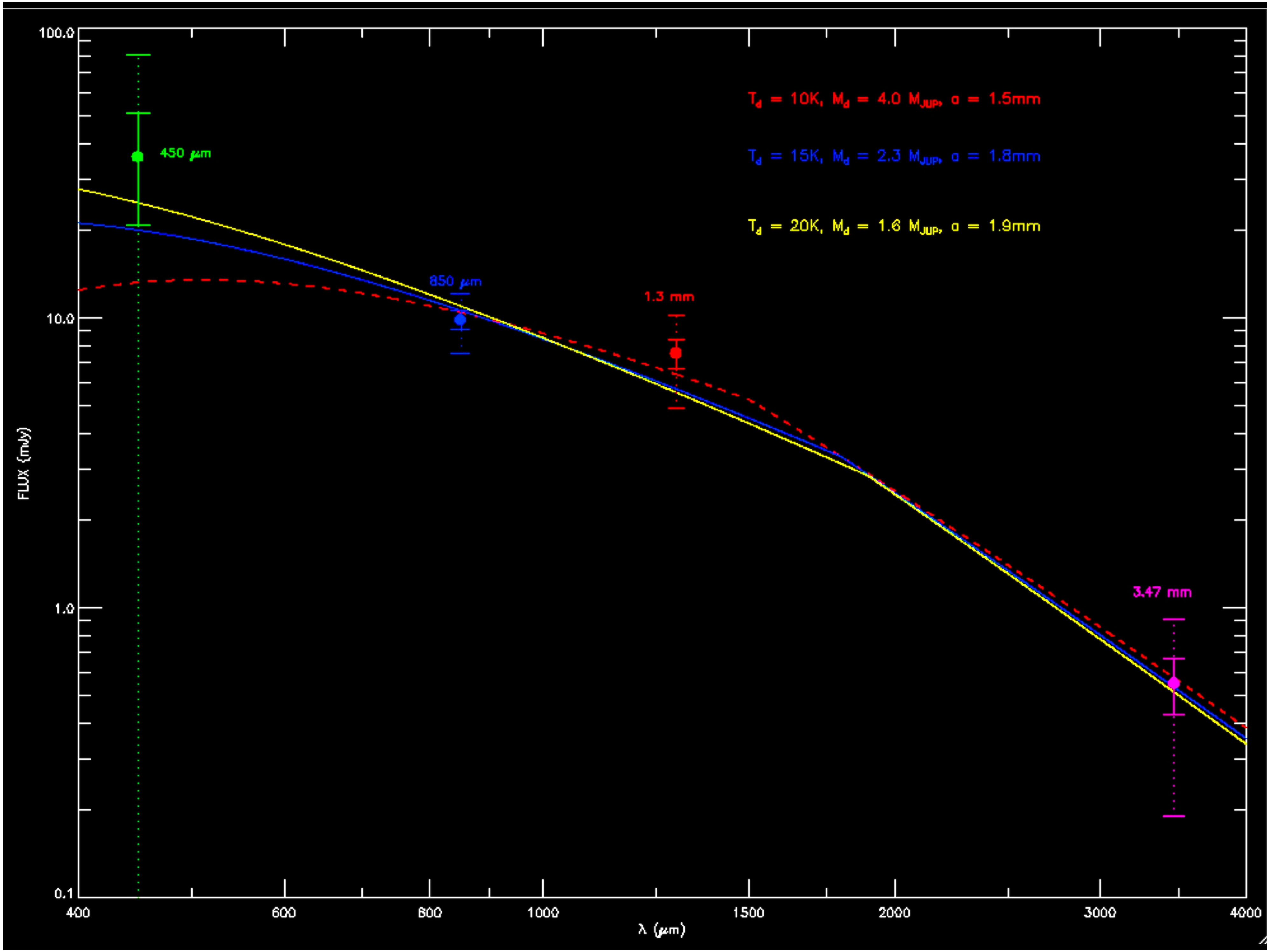
$M_d (t) \propto t^{-1/2}$

(Hartmann et al. 2006)

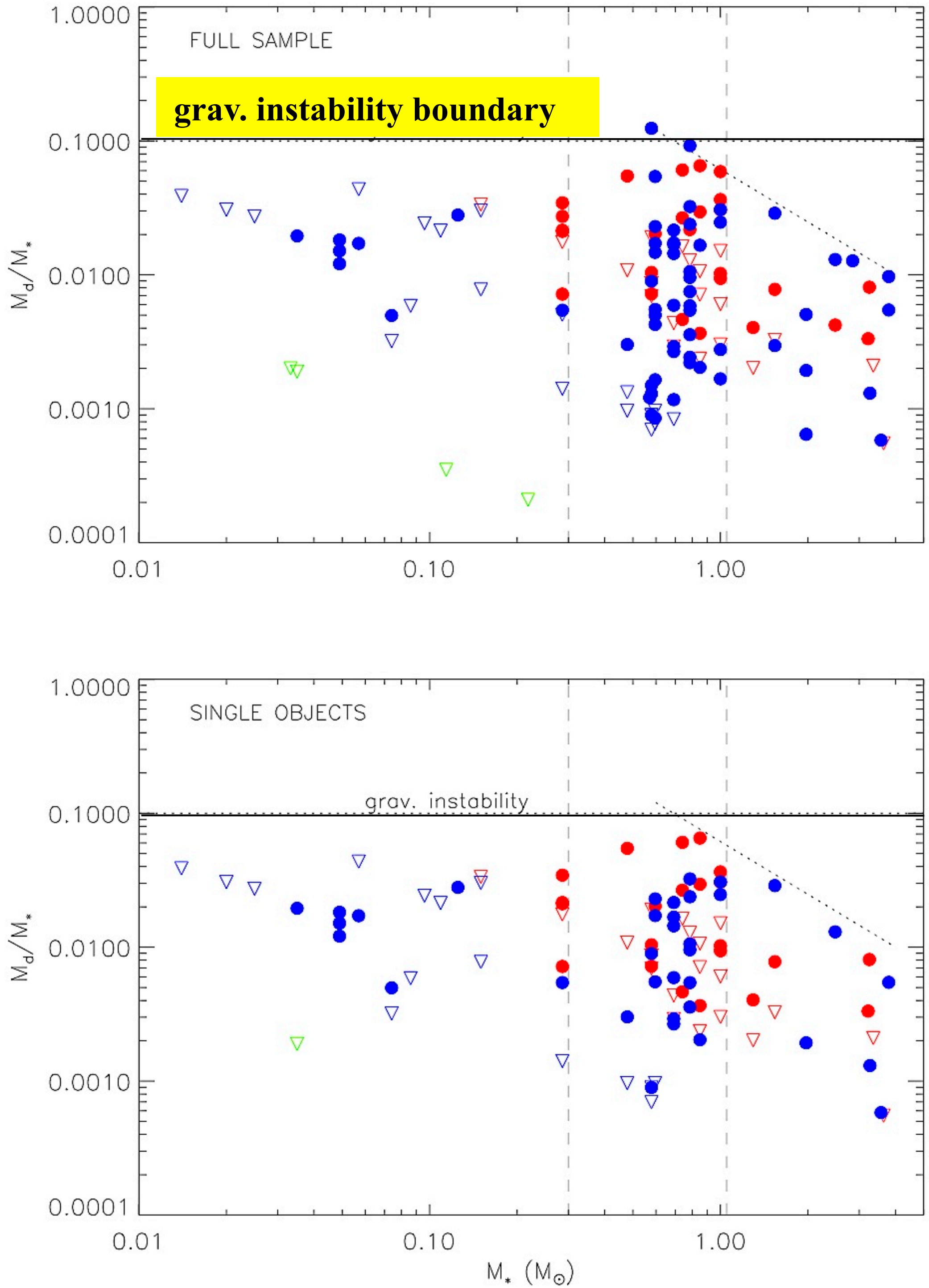
GRAIN GROWTH for a given M_* :

$M_d (t) \propto t^{-1}$

So accretion alone can explain change of TWA BDs from the Taurus mean, and grain growth alone can explain change of TWA VLMS from Taurus mean (growth required even if accretion invoked for latter sources).



T_d = 10K, M_d = 4.0 M_{JUP}, a = 1.5mm
T_d = 15K, M_d = 2.3 M_{JUP}, a = 1.8mm
T_d = 20K, M_d = 1.6 M_{JUP}, a = 1.9mm
450 μm
850 μm
1.3 mm
3.47 mm
FLUX (mJy)
λ (μm)
100.0
10.0
1.0
0.1
400
600
800
1000
1500
2000
3000
4000



Upper boundary in VLMS/BDs and solar-type cTTs close to grav. instability boundary (within a factor of 2-3).

why so close?

Hypothesis: grav. instabilities in *earlier* evolutionary epochs set this upper limit.

Grain growth (+ possibly lower T_d in VLMS/BDs) likely to increase disk mass estimate.

However, Ricci et al. (2010) show some evidence of more grain growth in fainter disks (which maybe why they're fainter), so may preferentially increase mass of our lower estimated disk masses, leaving upper boundary intact and moving *everything* closer to instability boundary, strengthening above hypothesis.

THE END