Secular evolution of hierarchical quadruple systems

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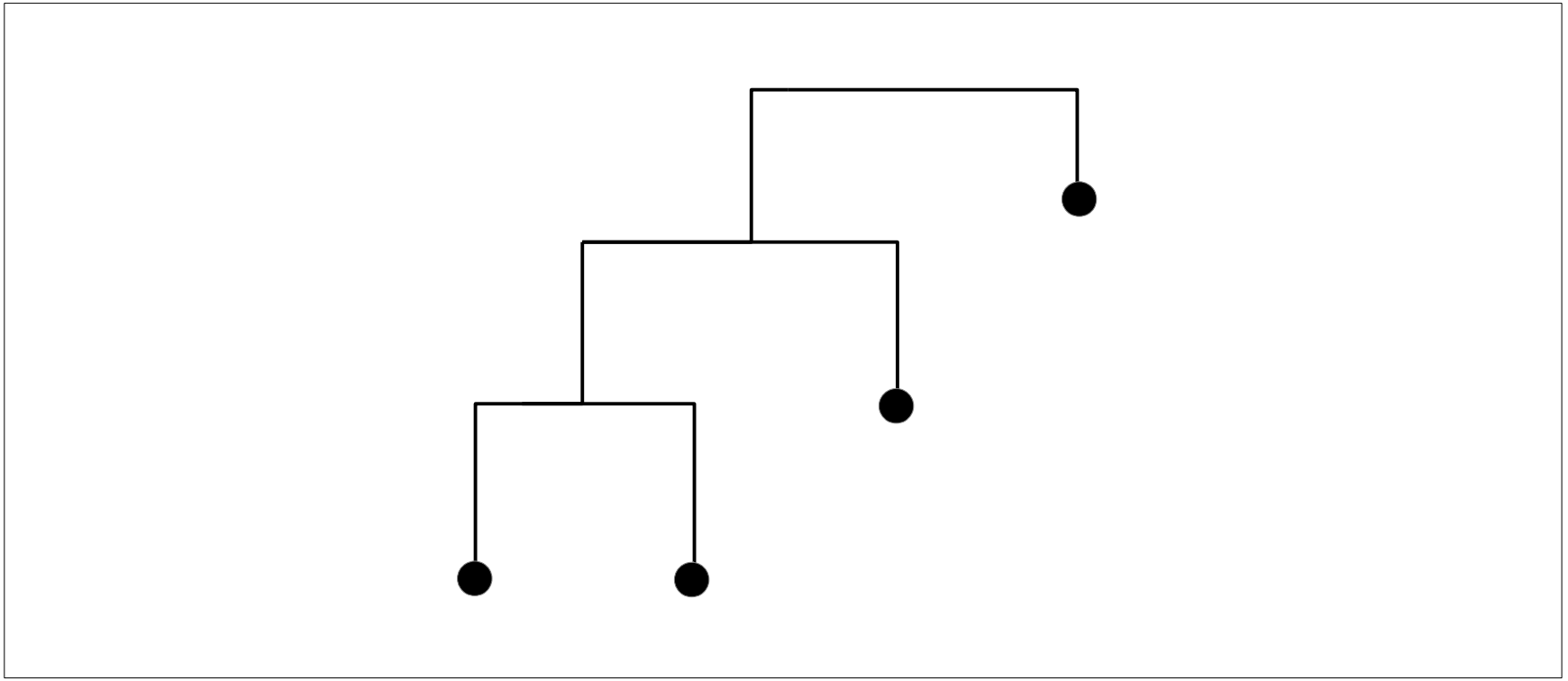

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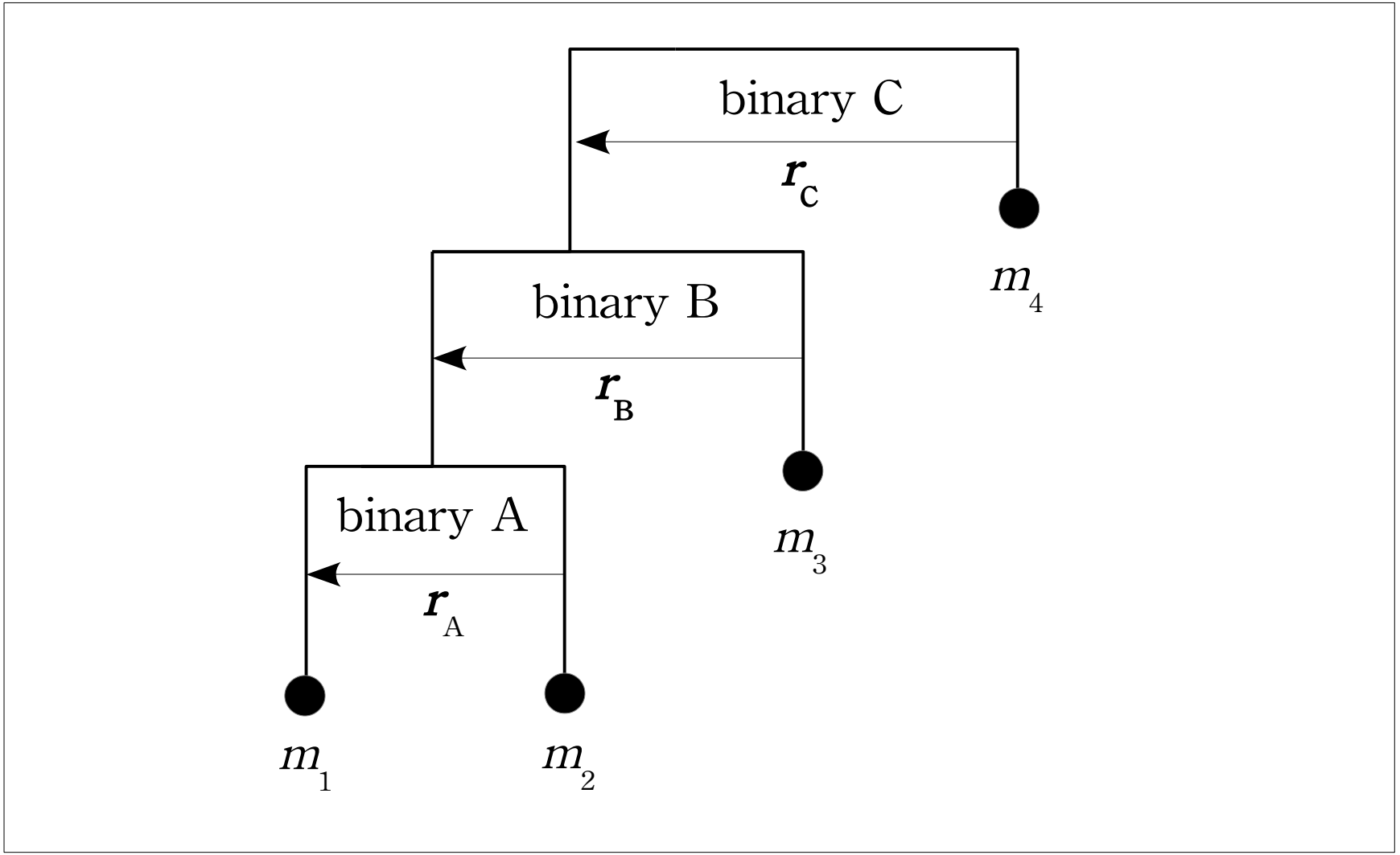
1. Introduction

Hierarchy



Assumption: $r_C \gg r_B \gg r_A$

Motivation

- 1% of observed F/G dwarf systems (d<67 pc) are quadruples (Tokovinin 2014)
  - ~ 1/3 in hierarchy considered here
  - fraction likely larger for more massive stars
- Dynamics in planetary systems
  - planet + moon around star, orbited by distant companion (e.g. Takeda+ 2008)
- Binaries orbiting SBH (e.g. Hopman 2009, Antonini+ 2010, 2012, Prodan+ 2014)
  - What happens if torqued by distant perturber (IBH, VRR, CND, ...)?

Triple dynamics

- Mutually orbiting & dynamically stable binaries A & B:
  - $i_{\rm AB} \gtrsim 39.2^\circ \rightarrow$ Kozai-Lidov (KL) cycles (Lidov 62, Kozai 62)
  - Eccentricity oscillations in A on time-scale

$$P_{\rm KL,AB} = \alpha \frac{P_{\rm B}^2}{P_{\rm A}} \frac{m_1 + m_2 + m_3}{m_3} \left(1 - e_{\rm B}^2\right)^{3/2}$$

  - Maximum eccentricity (simplest case)

$$e_{\rm A,max} = \sqrt{1 - \frac{5}{3} \cos^2 i_{\rm AB,0}}$$

  - Octupole terms $\rightarrow$ orbital flips $\rightarrow$ very high eccentricities
  - Additional precession of argument of periapse $\rightarrow$ quenching

2. Theory: Hamiltonian description

Hamiltonian

- $H = T + V = \frac{1}{2}\sum_{i=1}^{4} m_i \boldsymbol{V}_i^2 - \frac{1}{2}\sum_{i,j;i\neq j} \frac{G m_i m_j}{||\boldsymbol{R}_i - \boldsymbol{R}_j||}$

- Relative position vectors

  $\boldsymbol{r}_{\rm A} \equiv \boldsymbol{R}_1 - \boldsymbol{R}_2;$

  $\boldsymbol{r}_{\rm B} \equiv \frac{m_1\boldsymbol{R}_1 + m_2\boldsymbol{R}_2}{m_1+m_2} - \boldsymbol{R}_3;$

  $\boldsymbol{r}_{\rm C} \equiv \frac{m_1\boldsymbol{R}_1 + m_2\boldsymbol{R}_2 + m_3\boldsymbol{R}_3}{m_1+m_2+m_3} - \boldsymbol{R}_4.$

- Center of mass: $\dot{\boldsymbol{r}}_{\rm CM} = \boldsymbol{0}$

- Expand potential in $r_{\rm A}/r_{\rm C}$ and $r_{\rm B}/r_{\rm C}$

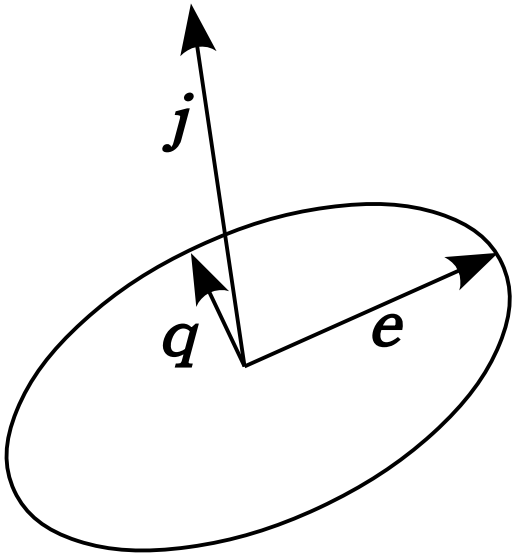
After some algebra...

- $H = H_{\rm bin,A} + H_{\rm bin,B} + H_{\rm bin,C}$
  $+ H_{\rm quad,AB} + H_{\rm quad,BC} + H_{\rm quad,AC}$
  $+ H_{\rm oct,AB} + H_{\rm oct,BC} + H_{\rm oct,AC} + H_{\rm oct,cross}$
  $+ H_{\rm hd,AB} + H_{\rm hd,BC} + H_{\rm hd,AC} + H_{\rm hd,cross,1} + H_{\rm hd,cross,2} + \cdots$

- $H_{\rm bin}$: binary Hamiltonian $\rightarrow$ $-Gmm'/(2a_k)$
- $H_{{\rm quad},kl}$, $H_{{\rm oct},kl}$, $H_{{\rm hd},kl}$: quadrupole, octupole and hexadecupole terms
  - mathematically equivalent to terms in N=3 problem
  - applied to all combinations AB, BC and AC
- $H_{\rm oct,cross}$, $H_{{\rm hd,cross},i}$: octupole & hexadecupole "cross" terms
  - depend on properties of **all** three binaries!

Orbit averaging

- *H* still a function of $\boldsymbol{r}_{\mathrm{A}}, \boldsymbol{r}_{\mathrm{B}}$ and r_{C}
- *Secular* approximation: average over r_k assuming unperturbed binary orbits
- Describe A, B & C orbits in terms of *orbital vectors* (**j**,**e**,**q**)



$$\boldsymbol{e} \cdot \boldsymbol{j} = 0$$

$$\boldsymbol{q} = \boldsymbol{j} \times \boldsymbol{e}$$

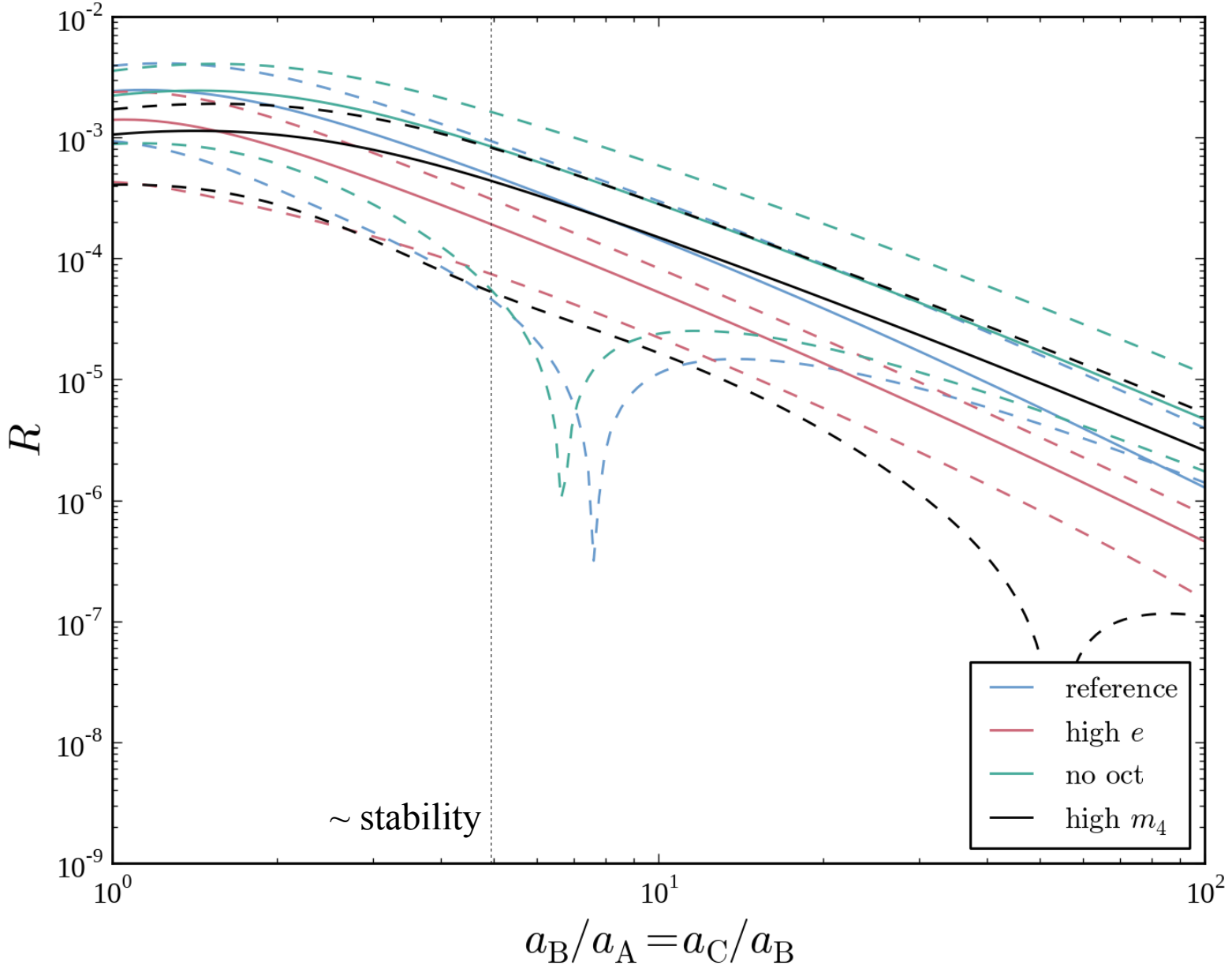
Result: cross term at octupole order

$$
\begin{aligned}
\overline{H}_{\mathrm{oct,\,cross}}(m_1,m_2,m_3,m_4,a_\mathrm{A},a_\mathrm{B},a_\mathrm{C},\boldsymbol{j}_\mathrm{A},\boldsymbol{e}_\mathrm{A},\boldsymbol{j}_\mathrm{B},\boldsymbol{e}_\mathrm{B},\boldsymbol{j}_\mathrm{C},\boldsymbol{e}_\mathrm{C}) = {}& \frac{Gm_1m_2m_3m_4}{(m_1+m_2)(m_1+m_2+m_3)}\frac{1}{a_\mathrm{C}}\left(\frac{a_\mathrm{A}}{a_\mathrm{C}}\right)^2\left(\frac{a_\mathrm{B}}{a_\mathrm{C}}\right)\frac{9}{32j_\mathrm{C}^5} \\
\times \Bigg[& 2\left(1-e_\mathrm{A}^2\right)\left(\boldsymbol{e}_\mathrm{B}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)\left(\boldsymbol{e}_\mathrm{C}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)\left\{4-5\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)^2\right\} - 10\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\boldsymbol{e}_\mathrm{C}\right)\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)\left\{\left(1+4e_\mathrm{A}^2\right)\left(\boldsymbol{e}_\mathrm{B}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)\right. \\
& \left. -\left(1-e_\mathrm{A}^2\right)\left(\boldsymbol{e}_\mathrm{B}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)\right\} + \left(\boldsymbol{e}_\mathrm{B}\cdot\boldsymbol{e}_\mathrm{C}\right)\Bigg\{-\left(1-6e_\mathrm{A}^2\right) - 10\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{e}}_\mathrm{C}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)^2 - 5\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)^2 \\
& \times\left[5e_\mathrm{A}^2 - 2\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{e}}_\mathrm{C}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)^2\right] - 20\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{e}}_\mathrm{C}\right)\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)\left(\hat{\boldsymbol{e}}_\mathrm{C}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right) + 5\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)^2 \\
& -10\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{e}}_\mathrm{C}\right)^2\left[1-\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)^2\right]\Bigg\} + 10\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\boldsymbol{e}_\mathrm{B}\right)\Bigg\{\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)\left(\boldsymbol{e}_\mathrm{C}\cdot\hat{\boldsymbol{j}}_\mathrm{A}\right)\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right) \\
& + \left(\hat{\boldsymbol{e}}_\mathrm{A}\cdot\boldsymbol{e}_\mathrm{C}\right)\left[1-\left(1-e_\mathrm{A}^2\right)\left(\hat{\boldsymbol{j}}_\mathrm{A}\cdot\hat{\boldsymbol{j}}_\mathrm{C}\right)^2\right]\Bigg\}\Bigg]
\end{aligned}
$$

Result: 2nd cross term at hexadecupole order

(-3*aA^2*aB^2*m1*m2*m3^2*m4*(-146 + 76*eAnorm^2 + 76*eBnorm^2 - 3656*eAnorm^2*eBnorm^2 - 309*eCnorm^2 + 54*eAnorm^2*eCnorm^2 + 54*eBnorm^2*eCnorm^2 - 1524*eAnorm^2*eBnorm^2*eCnorm^2 + 160*eAnorm^2*(eA . jB)^2 - 160*eAnorm^2*eBnorm^2*(eA . jB)^2 + 240*eAnorm^2*eCnorm^2*(eA . jB)^2 - 240*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jB)^2 - 50*eAnorm^2*(eA . jC)^2 + 4300*eAnorm^2*eBnorm^2*(eA . jC)^2 + 225*eAnorm^2*eCnorm^2*(eA . jC)^2 + 1650*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jC)^2 + 160*(eB . eC)^2 - 160*eAnorm^2*(eB . eC)^2 - 60*eBnorm^2*(eB . eC)^2 + 3560*eAnorm^2*eBnorm^2*(eB . eC)^2 + 360*eCnorm^2*(eB . eC)^2 - 360*eAnorm^2*eCnorm^2*(eB . eC)^2 - 60*eBnorm^2*eCnorm^2*(eB . eC)^2 + 1560*eAnorm^2*eBnorm^2*eCnorm^2*(eB . eC)^2 - 3500*eAnorm^2*eBnorm^2*(eA . jC)^2*(eB . eC)^2 + 160*eBnorm^2*(eB . jA)^2 - 160*eAnorm^2*eBnorm^2*(eB . jA)^2 + 240*eBnorm^2*eCnorm^2*(eB . jA)^2 - 240*eAnorm^2*eBnorm^2*eCnorm^2*(eB . jA)^2 - 50*eBnorm^2*(eB . jC)^2 + 4300*eAnorm^2*eBnorm^2*(eB . jC)^2 + 225*eBnorm^2*eCnorm^2*(eB . jC)^2 + 1650*eAnorm^2*eBnorm^2*eCnorm^2*(eB . jC)^2 - 5250*eAnorm^2*eBnorm^2*(eA . jC)^2*(eB . jC)^2 - 2625*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jC)^2*(eB . jC)^2 - 160*eB . eC*eB . jA*eC . jA + 160*eAnorm^2*eB . eC*eB . jA*eC . jA - 640*eBnorm^2*eB . eC*eB . jA*eC . jA + 640*eAnorm^2*eBnorm^2*eB . eC*eB . jA*eC . jA - 360*eCnorm^2*eB . eC*eB . jA*eC . jA + 360*eAnorm^2*eCnorm^2*eB . eC*eB . jA*eC . jA - 1440*eBnorm^2*eCnorm^2*eB . eC*eB . jA*eC . jA + 1440*eAnorm^2*eBnorm^2*eCnorm^2*eB . eC*eB . jA*eC . jA + 140*(eC . jA)^2 - 140*eAnorm^2*(eC . jA)^2 - 840*eBnorm^2*(eC . jA)^2 + 840*eAnorm^2*eBnorm^2*(eC . jA)^2 + 300*eCnorm^2*(eC . jA)^2 - 300*eAnorm^2*eCnorm^2*(eC . jA)^2 - 600*eBnorm^2*eCnorm^2*(eC . jA)^2 + 600*eAnorm^2*eBnorm^2*eCnorm^2*(eC . jA)^2 + 1400*eBnorm^2*(eB . eC)^2*(eC . jA)^2 - 1400*eAnorm^2*eBnorm^2*(eB . eC)^2*(eC . jA)^2 + 2100*eBnorm^2*eCnorm^2*(eB . eC)^2*(eC . jA)^2 - 2100*eAnorm^2*eBnorm^2*eCnorm^2*(eB . eC)^2*(eC . jA)^2 + 700*eBnorm^2*(eB . jC)^2*(eC . jA)^2 - 700*eAnorm^2*eBnorm^2*(eB . jC)^2*(eC . jA)^2 + 10*(eC . jB)^2 - 860*eAnorm^2*(eC . jB)^2 - 10*eBnorm^2*(eC . jB)^2 + 860*eAnorm^2*eBnorm^2*(eC . jB)^2 - 45*eCnorm^2*(eC . jB)^2 - 330*eAnorm^2*eCnorm^2*(eC . jB)^2 + 45*eBnorm^2*eCnorm^2*(eC . jB)^2 + 330*eAnorm^2*eBnorm^2*eCnorm^2*(eC . jB)^2 + 1050*eAnorm^2*(eA . jC)^2*(eC . jB)^2 - 1050*eAnorm^2*eBnorm^2*(eA . jC)^2*(eC . jB)^2 + 525*eAnorm^2*eCnorm^2*(eA . jC)^2*(eC . jB)^2 - 525*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jC)^2*(eC . jB)^2 + 140*eAnorm^2*(eC . jA)^2*(eC . jB)^2 + 140*eBnorm^2*(eC . jA)^2*(eC . jB)^2 - 140*eAnorm^2*eBnorm^2*(eC . jA)^2*(eC . jB)^2 - 32*(jA . jB)^2 + 32*eAnorm^2*(jA . jB)^2 + 32*eBnorm^2*(jA . jB)^2 - 32*eAnorm^2*eBnorm^2*(jA . jB)^2 - 48*eCnorm^2*(jA . jB)^2 + 48*eAnorm^2*eCnorm^2*(jA . jB)^2 + 48*eBnorm^2*eCnorm^2*(jA . jB)^2 - 48*eAnorm^2*eBnorm^2*eCnorm^2*(jA . jB)^2 + 40*(eA . eB)^2*(4 + 16*eAnorm^2 + 16*eBnorm^2 + 44*eAnorm^2*eBnorm^2 + 3*eCnorm^2 + 12*eAnorm^2*eCnorm^2 + 12*eBnorm^2*eCnorm^2 + 18*eAnorm^2*eBnorm^2*eCnorm^2 + (-1 + eAnorm^2)*(-1 + eBnorm^2)*(4 + 9*eCnorm^2)*eC . jA*eC . jB*jA . jB - (-1 + eAnorm^2)*(-1 + eBnorm^2)*(4 + 9*eCnorm^2)*(jA . jB)^2) + 10*(jA . jC)^2 - 10*eAnorm^2*(jA . jC)^2 - 860*eBnorm^2*(jA . jC)^2 + 860*eAnorm^2*eBnorm^2*(jA . jC)^2 - 45*eCnorm^2*(jA . jC)^2 + 45*eAnorm^2*eCnorm^2*(jA . jC)^2 - 330*eBnorm^2*eCnorm^2*(jA . jC)^2 + 330*eAnorm^2*eBnorm^2*eCnorm^2*(jA . jC)^2 + 700*eBnorm^2*(eB . eC)^2*(jA . jC)^2 - 700*eAnorm^2*eBnorm^2*(eB . eC)^2*(jA . jC)^2 + 1050*eBnorm^2*(eB . jC)^2*(jA . jC)^2 - 1050*eAnorm^2*eBnorm^2*(eB . jC)^2*(jA . jC)^2 + 525*eBnorm^2*eCnorm^2*(eB . jC)^2*(jA . jC)^2 - 525*eAnorm^2*eBnorm^2*eCnorm^2*(eB . jC)^2*(jA . jC)^2 - 210*(eC . jB)^2*(jA . jC)^2 + 210*eAnorm^2*(eC . jB)^2*(jA . jC)^2 + 210*eBnorm^2*(eC . jB)^2*(jA . jC)^2 - 210*eAnorm^2*eBnorm^2*(eC . jB)^2*(jA . jC)^2 - 105*eCnorm^2*(eC . jB)^2*(jA . jC)^2 + 105*eAnorm^2*eCnorm^2*(eC . jB)^2*(jA . jC)^2 + 105*eBnorm^2*eCnorm^2*(eC . jB)^2*(jA . jC)^2 - 105*eAnorm^2*eBnorm^2*eCnorm^2*(eC . jB)^2*(jA . jC)^2 + 130*eC . jB*jB . eC + 20*eAnorm^2*eC . jB*jB . eC - 130*eBnorm^2*eC . jB*jB . eC - 20*eAnorm^2*eBnorm^2*eC . jB*jB . eC + 345*eCnorm^2*eC . jB*jB . eC - 270*eAnorm^2*eCnorm^2*eC . jB*jB . eC - 345*eBnorm^2*eCnorm^2*eC . jB*jB . eC + 270*eAnorm^2*eBnorm^2*eCnorm^2*eC . jB*jB . eC - 350*eAnorm^2*(eA . jC)^2*eC . jB*jB . eC + 350*eAnorm^2*eBnorm^2*(eA . jC)^2*eC . jB*jB . eC - 525*eAnorm^2*eCnorm^2*(eA . jC)^2*eC . jB*jB . eC + 525*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jC)^2*eC . jB*jB . eC - 140*(eC . jA)^2*eC . jB*jB . eC + 140*eAnorm^2*(eC . jA)^2*eC . jB*jB . eC - 140*eAnorm^2*eBnorm^2*(eC . jA)^2*eC . jB*jB . eC - 420*eCnorm^2*(eC . jA)^2*eC . jB*jB . eC + 420*eAnorm^2*eCnorm^2*(eC . jA)^2*eC . jB*jB . eC + 420*eBnorm^2*eCnorm^2*(eC . jA)^2*eC . jB*jB . eC - 420*eAnorm^2*eBnorm^2*eCnorm^2*(eC . jA)^2*eC . jB*jB . eC + 70*eC . jB*(jA . jC)^2*jB . eC - 70*eAnorm^2*eC . jB*(jA . jC)^2*jB . eC - 70*eBnorm^2*eC . jB*(jA . jC)^2*jB . eC + 70*eAnorm^2*eBnorm^2*eC . jB*(jA . jC)^2*jB . eC + 105*eCnorm^2*eC . jB*(jA . jC)^2*jB . eC - 105*eAnorm^2*eCnorm^2*eC . jB*(jA . jC)^2*jB . eC - 105*eBnorm^2*eCnorm^2*eC . jB*(jA . jC)^2*jB . eC + 105*eAnorm^2*eBnorm^2*eCnorm^2*eC . jB*(jA . jC)^2*jB . eC + 160*eA . jB*eA . jC*(eB . eC)^2*jB . jC + 640*eAnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC - 160*eBnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC - 640*eAnorm^2*eBnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC + 120*eCnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC + 480*eAnorm^2*eCnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC - 120*eBnorm^2*eCnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC - 480*eAnorm^2*eBnorm^2*eCnorm^2*eA . jB*eA . jC*(eB . eC)^2*jB . jC + 160*eB . eC*eB . jC*jB . eC*jB . jC + 640*eAnorm^2*eB . eC*eB . jC*jB . eC*jB . jC - 160*eBnorm^2*eB . eC*eB . jC*jB . eC*jB . jC - 640*eAnorm^2*eBnorm^2*eB . eC*eB . jC*jB . eC*jB . jC + 120*eCnorm^2*eB . eC*eB . jC*jB . eC*jB . jC + 480*eAnorm^2*eCnorm^2*eB . eC*eB . jC*jB . eC*jB . jC - 120*eBnorm^2*eCnorm^2*eB . eC*eB . jC*jB . eC*jB . jC - 480*eAnorm^2*eBnorm^2*eCnorm^2*eB . eC*eB . jC*jB . eC*jB . jC + 10*(jB . jC)^2 - 860*eAnorm^2*(jB . jC)^2 - 10*eBnorm^2*(jB . jC)^2 + 860*eAnorm^2*eBnorm^2*(jB . jC)^2 - 45*eCnorm^2*(jB . jC)^2 - 330*eAnorm^2*eCnorm^2*(jB . jC)^2 + 45*eBnorm^2*eCnorm^2*(jB . jC)^2 + 330*eAnorm^2*eBnorm^2*eCnorm^2*(jB . jC)^2 + 1050*eAnorm^2*(eA . jC)^2*(jB . jC)^2 - 1050*eAnorm^2*eBnorm^2*(eA . jC)^2*(jB . jC)^2 + 525*eAnorm^2*eCnorm^2*(eA . jC)^2*(jB . jC)^2 - 525*eAnorm^2*eBnorm^2*eCnorm^2*(eA . jC)^2*(jB . jC)^2 - 160*(eB . eC)^2*(jB . jC)^2 - 640*eAnorm^2*(eB . eC)^2*(jB . jC)^2 + 160*eBnorm^2*(eB . eC)^2*(jB . jC)^2 + 640*eAnorm^2*eBnorm^2*(eB . eC)^2*(jB . jC)^2 - 120*eCnorm^2*(eB . eC)^2*(jB . jC)^2 - 480*eAnorm^2*eCnorm^2*(eB . eC)^2*(jB . jC)^2 + 120*eBnorm^2*eCnorm^2*(eB . eC)^2*(jB . jC)^2 + 480*eAnorm^2*eBnorm^2*eCnorm^2*(eB . eC)^2*(jB . jC)^2 - 140*(eC . jA)^2*(jB . jC)^2 + 140*eAnorm^2*(eC . jA)^2*(jB . jC)^2 + 140*eBnorm^2*(eC . jA)^2*(jB . jC)^2 - 140*eAnorm^2*eBnorm^2*(eC . jA)^2*(jB . jC)^2 - 210*(jA . jC)^2*(jB . jC)^2 + 210*eAnorm^2*(jA . jC)^2*(jB . jC)^2 + 210*eBnorm^2*(jA . jC)^2*(jB . jC)^2 - 210*eAnorm^2*eBnorm^2*(jA . jC)^2*(jB . jC)^2 - 105*eCnorm^2*(jA . jC)^2*(jB . jC)^2 + 105*eAnorm^2*eCnorm^2*(jA . jC)^2*(jB . jC)^2 + 105*eBnorm^2*eCnorm^2*(jA . jC)^2*(jB . jC)^2 - 105*eAnorm^2*eBnorm^2*eCnorm^2*(jA . jC)^2*(jB . jC)^2 + 20*eA . eC*((1 + 4*eBnorm^2)*eA . jC*(-2*(-1 + eAnorm^2)*(4 + 3*eCnorm^2)*eC . jA*jA . jC + 7*(-1 + eAnorm^2)*(2 + 3*eCnorm^2)*(eB . eC)^2*eC . jA*jA . jC + eB . eC*(7*(1 + 4*eAnorm^2)*(2 + 3*eCnorm^2)*eB . jC - 3*(-1 + eAnorm^2)*(2 + 5*eCnorm^2)*eB . jA*jA . jC)) + (-1 + eBnorm^2)*eA . jB*(2*(1 + 4*eAnorm^2)*(4 + 9*eCnorm^2)*jB . eC - eB . eC*(3*(1 + 4*eAnorm^2)*(2 + 5*eCnorm^2)*eB . jC - 7*(-1 + eAnorm^2)*(2 + 3*eCnorm^2)*(eB . jA - eB . eC*eC . jA)*jA . jC)*jB . jC)) - 20*(eA . eC)^2*(-8 + 3*eAnorm^2 + 8*eBnorm^2 - 178*eAnorm^2*eBnorm^2 - 18*eCnorm^2 + 3*eAnorm^2*eCnorm^2 + 18*eBnorm^2*eCnorm^2 - 78*eAnorm^2*eBnorm^2*eCnorm^2 + 175*eAnorm^2*eBnorm^2*(eB . jC)^2 - 35*eAnorm^2*(eC . jB)^2 + 35*eAnorm^2*eBnorm^2*(eC . jB)^2 - 8*eB . jA*eB . jC*jA . jC + 8*eAnorm^2*eB . jA*eB . jC*jA . jC - 32*eBnorm^2*eB . jA*eB . jC*jA . jC + 32*eAnorm^2*eBnorm^2*eB . jA*eB . jC*jA . jC - 6*eCnorm^2*eB . jA*eB . jC*jA . jC + 6*eAnorm^2*eCnorm^2*eB . jA*eB . jC*jA . jC - 24*eBnorm^2*eCnorm^2*eB . jA*eB . jC*jA . jC + 24*eAnorm^2*eBnorm^2*eCnorm^2*eB . jA*eB . jC*jA . jC + 8*(jA . jC)^2 - 8*eAnorm^2*(jA . jC)^2 + 32*eBnorm^2*(jA . jC)^2 - 32*eAnorm^2*eBnorm^2*(jA . jC)^2 + 6*eCnorm^2*(jA . jC)^2 - 6*eAnorm^2*eCnorm^2*(jA . jC)^2 + 24*eBnorm^2*eCnorm^2*(jA . jC)^2 - 24*eAnorm^2*eBnorm^2*eCnorm^2*(jA . jC)^2 - 35*eAnorm^2*eC . jB*jB . eC + 35*eAnorm^2*eBnorm^2*eC . jB*jB . eC - 105*eAnorm^2*eCnorm^2*eC . jB*jB . eC + 105*eAnorm^2*eBnorm^2*eCnorm^2*eC . jB*jB . eC - 7*(-1 + eBnorm^2)*(2 + 3*eCnorm^2)*eB . eC*eC . jB*((1 + 4*eAnorm^2)*eB . jC - (-1 + eAnorm^2)*eB . jA*jA . jC)*jB . jC - 35*eAnorm^2*(jB . jC)^2 + 35*eAnorm^2*eBnorm^2*(jB . jC)^2 + 7*(2 + 3*eCnorm^2)*(eB . eC)^2*(-1 - 4*eAnorm^2 - 4*eBnorm^2 + 9*eAnorm^2*eBnorm^2 + (-1 + eAnorm^2)*(1 + 4*eBnorm^2)*(jA . jC)^2 - (-1 + eAnorm^2)*(-1 + eBnorm^2)*jA . jB*jA . jC*jB . jC + (1 + 4*eAnorm^2)*(-1 + eBnorm^2)*jB . jC*jC . jB)) + 20*eA . eB*(-2*((1 - eAnorm^2)*(-1 + eBnorm^2)*(4 + 9*eCnorm^2)*eA . jB*(eB . jA - eB . eC*eC . jA)*jA . jB + (1 + 4*eAnorm^2)*(4 + 3*eCnorm^2)*eA . jC*((1 + 4*eBnorm^2)*eB . jC - (-1 + eBnorm^2)*eB . eC*jB . eC*jB . jC)) + eA . eC*(-2*(-1 + eAnorm^2)*((-1 + eBnorm^2)*(4 + 9*eCnorm^2)*eB . jA*eC . jB*jA . jB - (1 + 4*eBnorm^2)*(4 + 3*eCnorm^2)*eB . jC*eC . jA*jA . jC) + eB . eC*(-30 - 80*eAnorm^2 - 80*eBnorm^2 - 160*eAnorm^2*eBnorm^2 - 45*eCnorm^2 - 90*eAnorm^2*eCnorm^2 - 90*eBnorm^2*eCnorm^2 + 2*(-1 + eAnorm^2)*(-1 + eBnorm^2)*(4 + 9*eCnorm^2)*(jA . jB)^2 + 3*(-1 + eAnorm^2)*(1 + 4*eBnorm^2)*(2 + 5*eCnorm^2)*(jA . jC)^2 + 7*(-1 + eAnorm^2)*(-1 + eBnorm^2)*(2 + 3*eCnorm^2)*eC . jA*eC . jB*jA . jC*jB . jC - (-1 + eAnorm^2)*(-1 + eBnorm^2)*(22 + 27*eCnorm^2)*jA . jB*jA . jC*jB . jC + 8*(jB . jC)^2 + 32*eAnorm^2*(jB . jC)^2 - 8*eBnorm^2*(jB . jC)^2 - 32*eAnorm^2*eBnorm^2*(jB . jC)^2 + 6*eCnorm^2*(jB . jC)^2 + 24*eAnorm^2*eCnorm^2*(jB . jC)^2 - 6*eBnorm^2*eCnorm^2*(jB . jC)^2 - 24*eAnorm^2*eBnorm^2*eCnorm^2*(jB . jC)^2 - 14*jB . jC*jC . jB - 56*eAnorm^2*jB . jC*jC . jB + 14*eBnorm^2*jB . jC*jC . jB + 56*eAnorm^2*eBnorm^2*jB . jC*jC . jB - 21*eCnorm^2*jB . jC*jC . jB - 84*eAnorm^2*eCnorm^2*jB . jC*jC . jB + 21*eBnorm^2*eCnorm^2*jB . jC*jC . jB + 84*eAnorm^2*eBnorm^2*eCnorm^2*jB . jC*jC . jB)))))/(256*aC^5*(1 - eCnorm^2)^(7/2)*(m1 + m2)*(m1 + m2 + m3)^2)

How important are the octupole order cross terms?



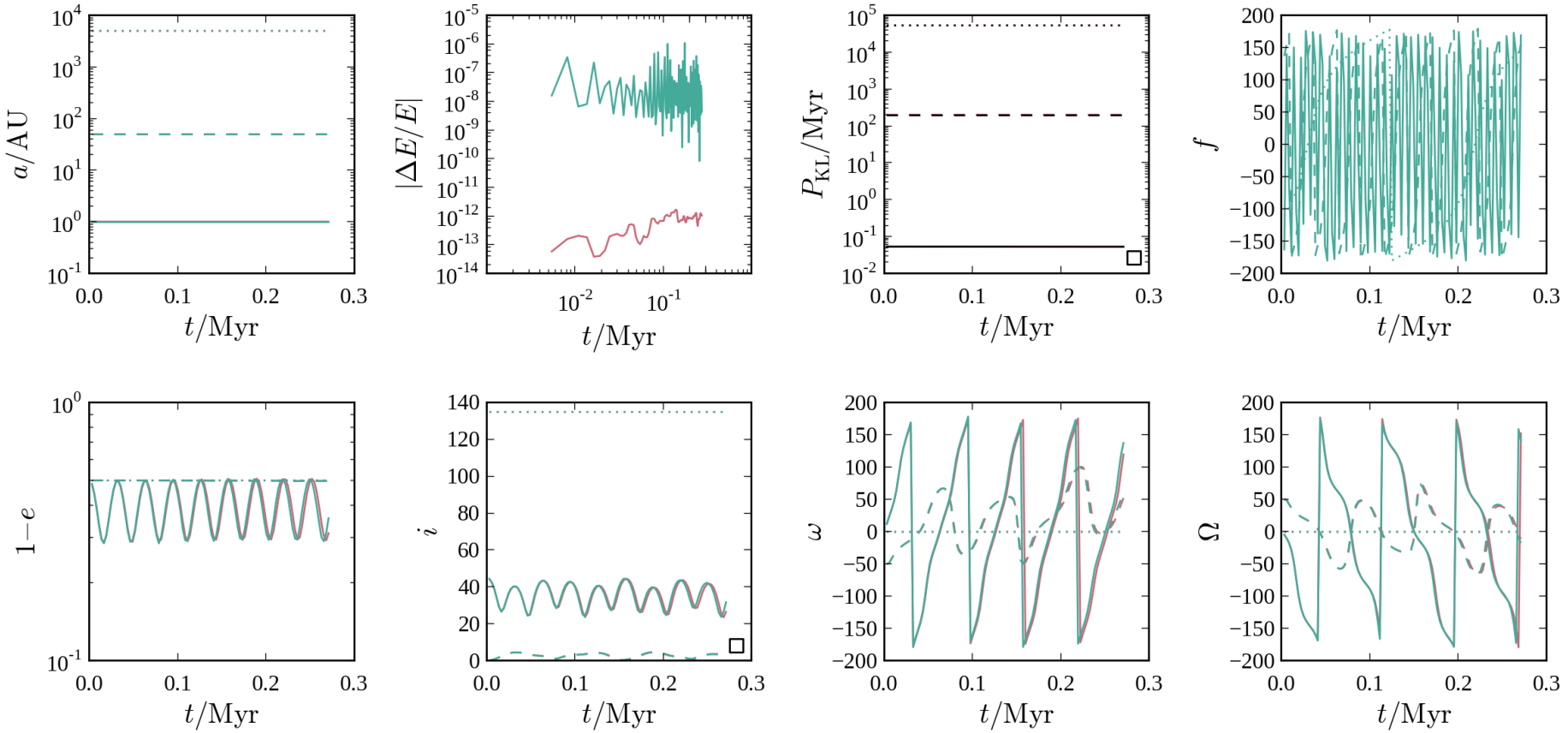
3. Numerical algorithm

secularquadruple

- Solves equations of motion from Hamiltonian
- System of ODEs
- Stiff equations; uses CVODE library (Cohen & Hindmarsh 1996)
- Includes GR effects: 1PN precession & 2.5PN GW dissipation
- Interfaced within AMUSE framework (***www.amusecode.org***)
  - Easy to combine with *N*-body codes, stellar and binary evolution, hydrodynamics, ...
- Very easy to switch between secular and direct *N*-body:

  "`--integration_method secular`" → "`--integration_method nbody`"
- Will (at some time) be part of publicly available community codes in AMUSE

Comparison to *N*-body

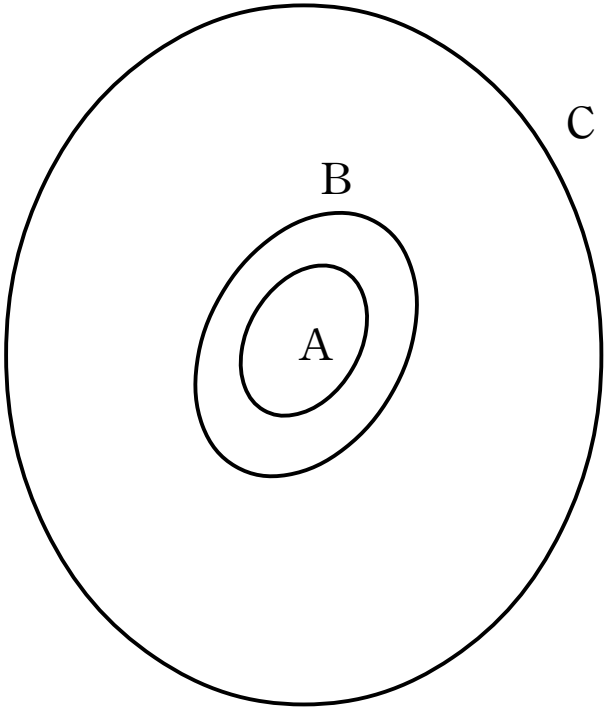


Direct *N*-body code (*Mikkola*)

secularquadruple (octupole order; no cross terms)

3. Torquing coplanar triples

Two extreme cases (not drawn to scale)

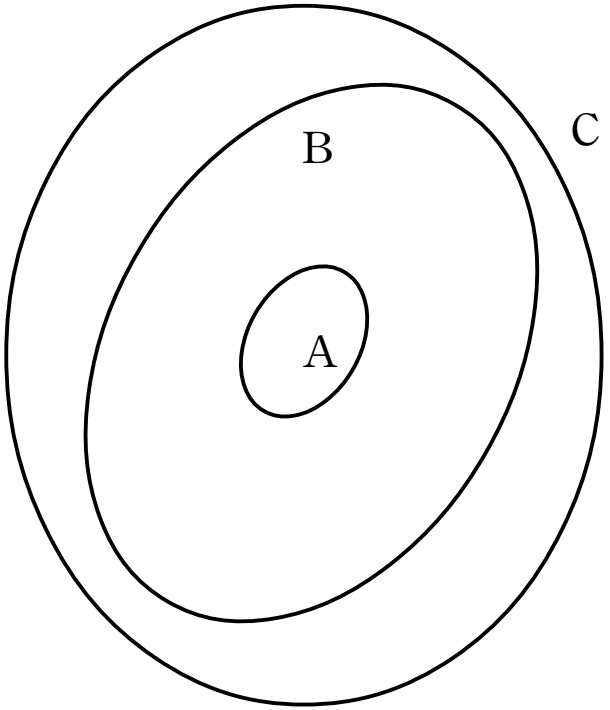


$$P_{\mathrm{KL,AB}} \ll P_{\mathrm{KL,BC}}$$

$$\mathcal{R} \equiv P_{\mathrm{KL,AB}}/P_{\mathrm{KL,BC}} \ll 1$$

Torque of B on A dominates

→ A & B remain coplanar



$$P_{\mathrm{KL,AB}} \gg P_{\mathrm{KL,BC}}$$

$$\mathcal{R} \equiv P_{\mathrm{KL,AB}}/P_{\mathrm{KL,BC}} \gg 1$$

Torque of C on B dominates

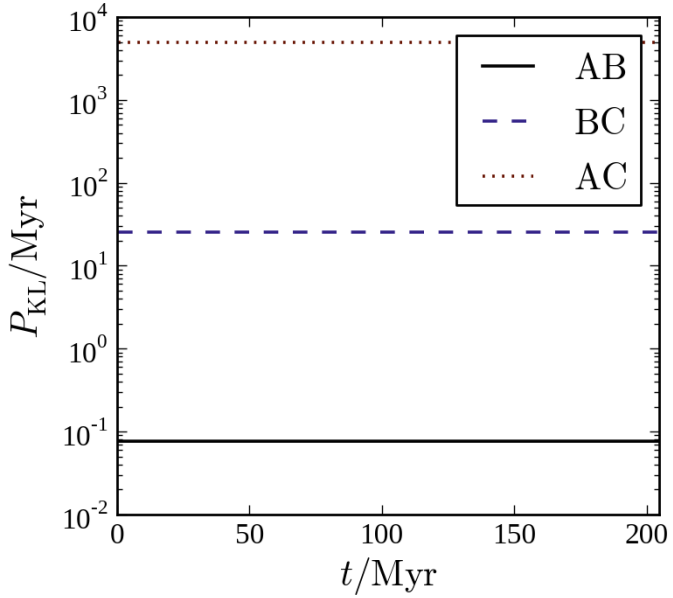
→ B & C change inclination

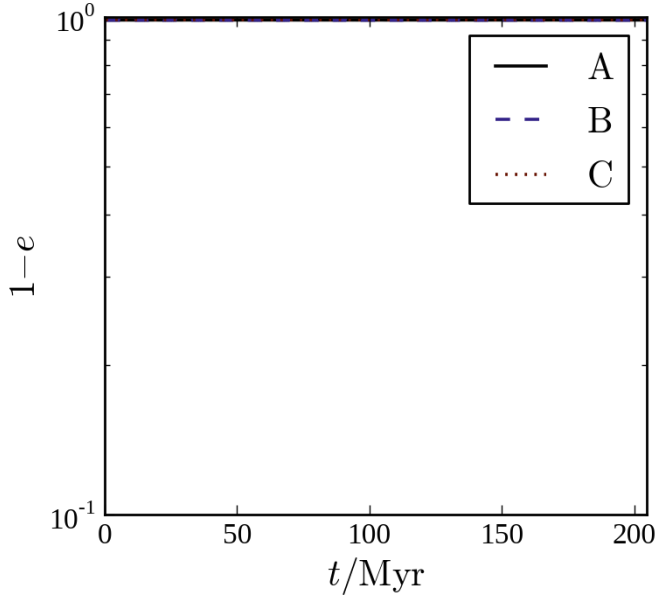
→ A & B become inclined

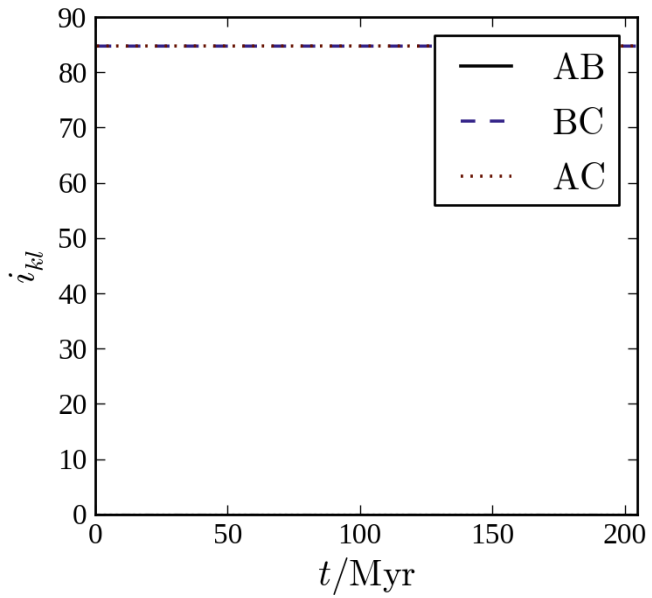
Examples

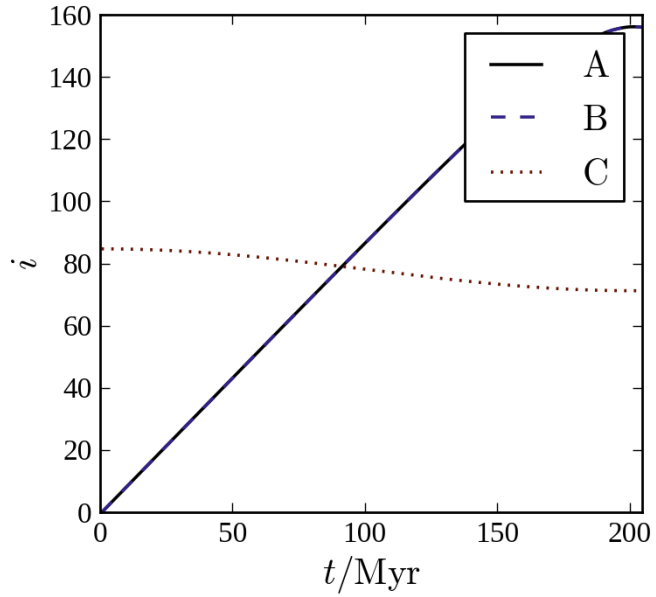
- A & B coplanar; inclined w.r.t. C with $i_{\rm AC} = i_{\rm BC} = 85^\circ$
- Initial eccentricities $e_{\rm A} = e_{\rm B} = e_{\rm C} = 0.01$
- Fixed $a_{\rm B} = 100$ AU; $a_{\rm C} = 5000$ AU
- Vary $a_{\rm A}$, therefore change $\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \propto a_{\rm A}^{-3/2}$

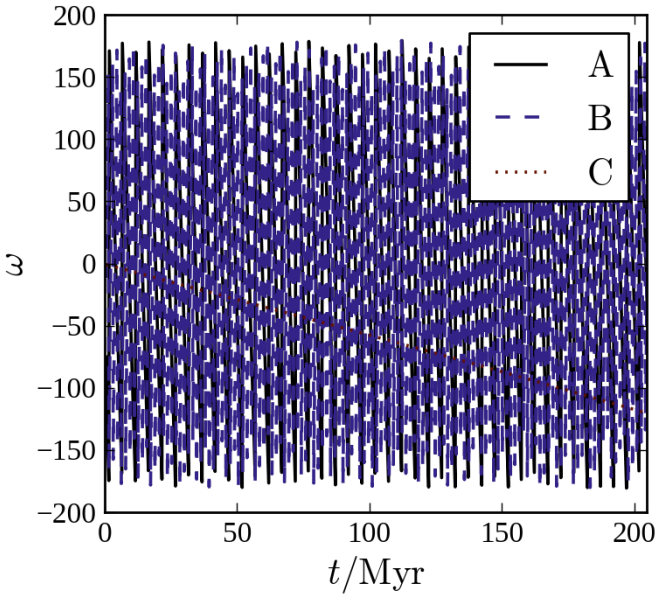
$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 3 \times 10^{-3}$$

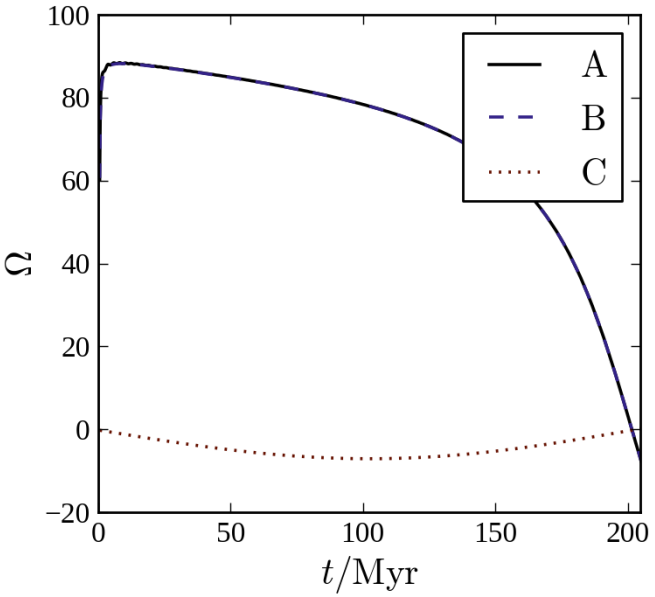




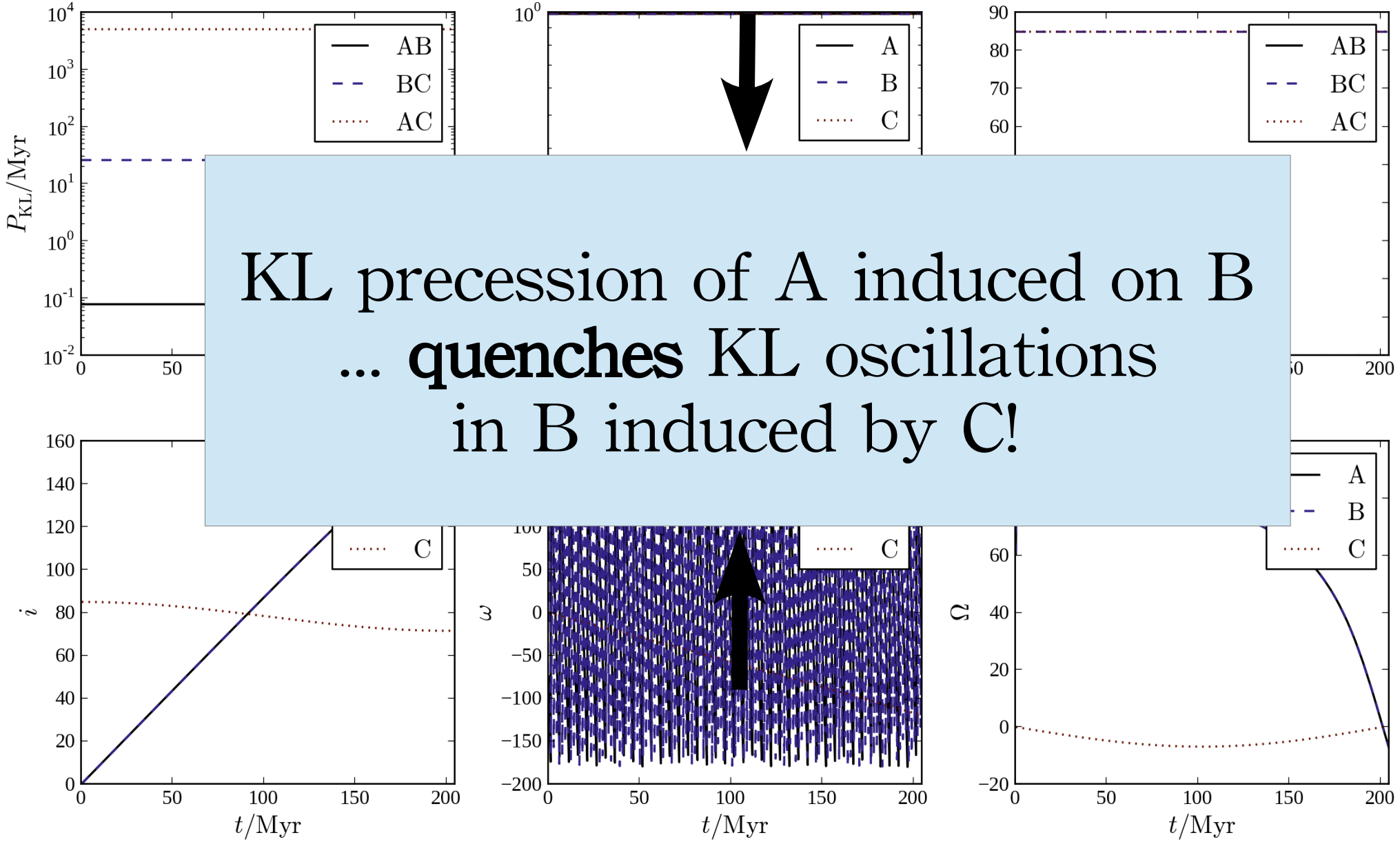






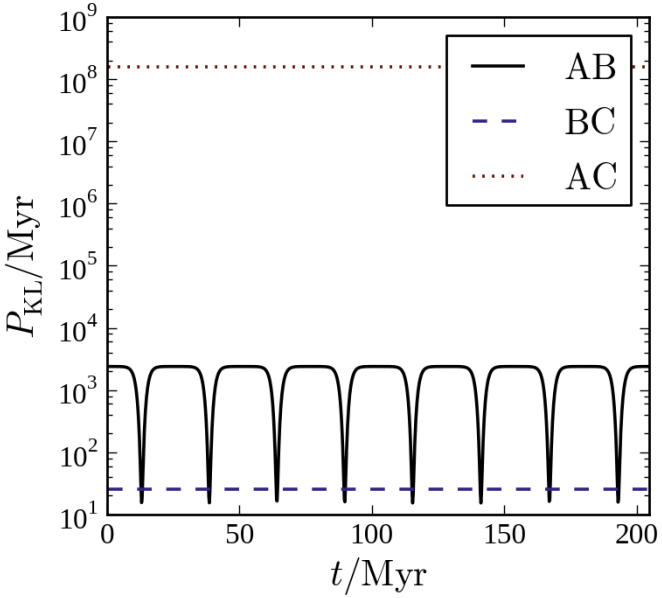


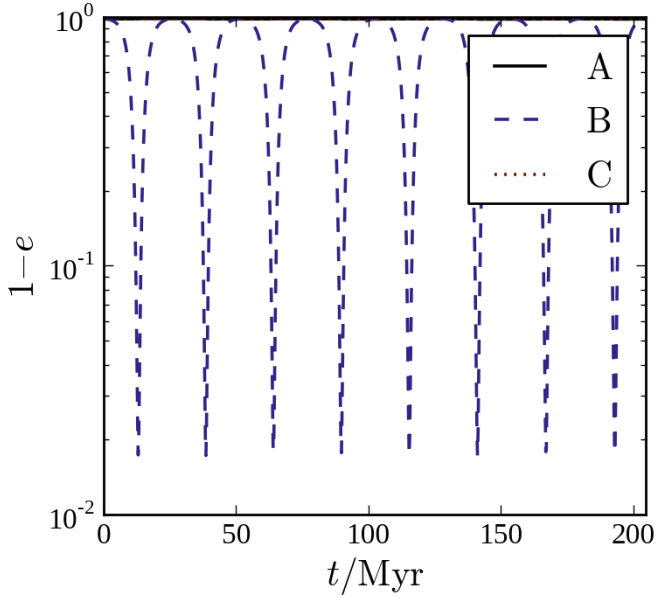
$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 3 \times 10^{-3}$$

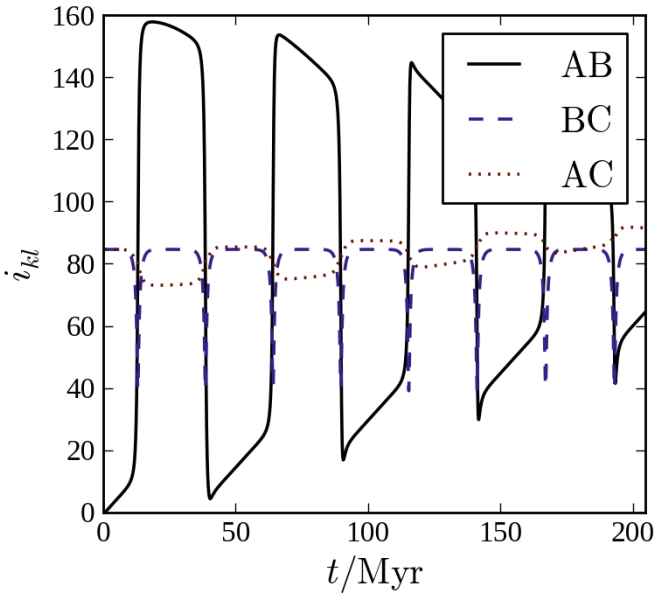


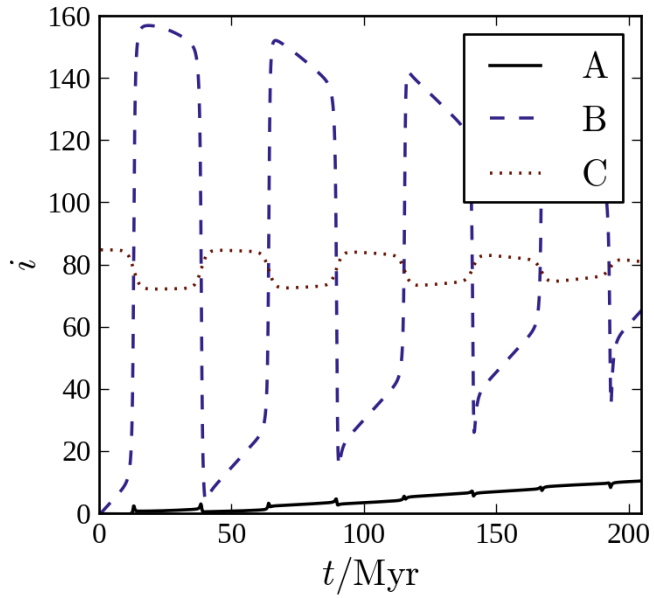
KL precession of A induced on B
... **quenches** KL oscillations
in B induced by C!

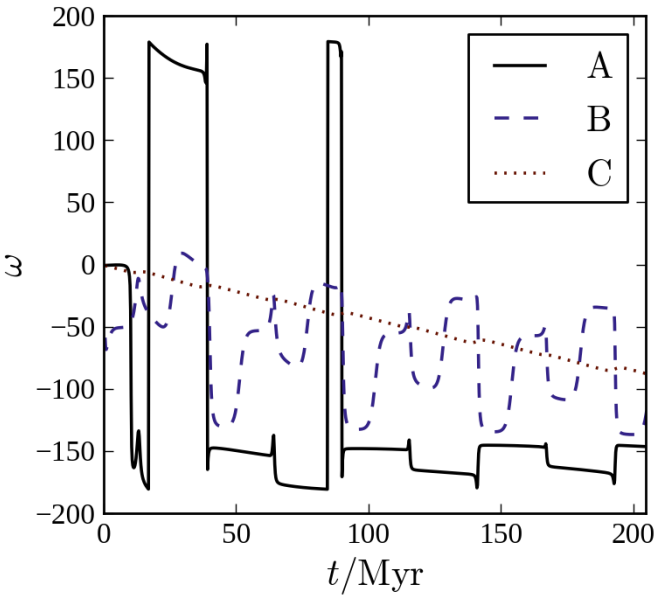
$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 1 \times 10^2$$

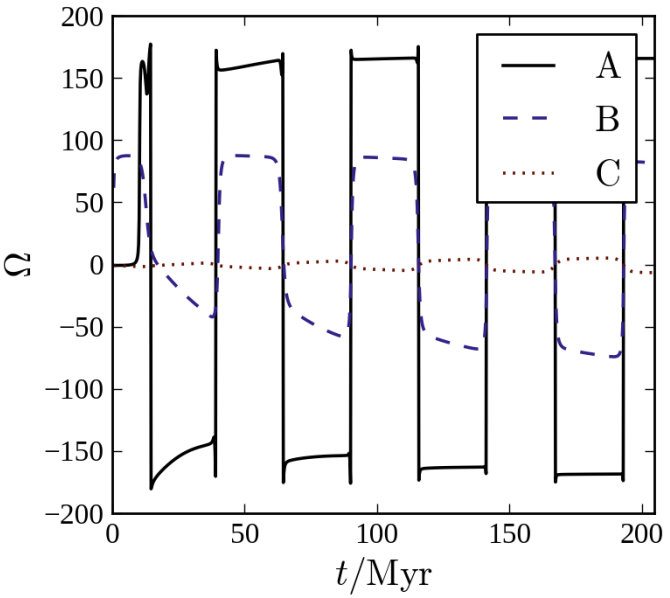




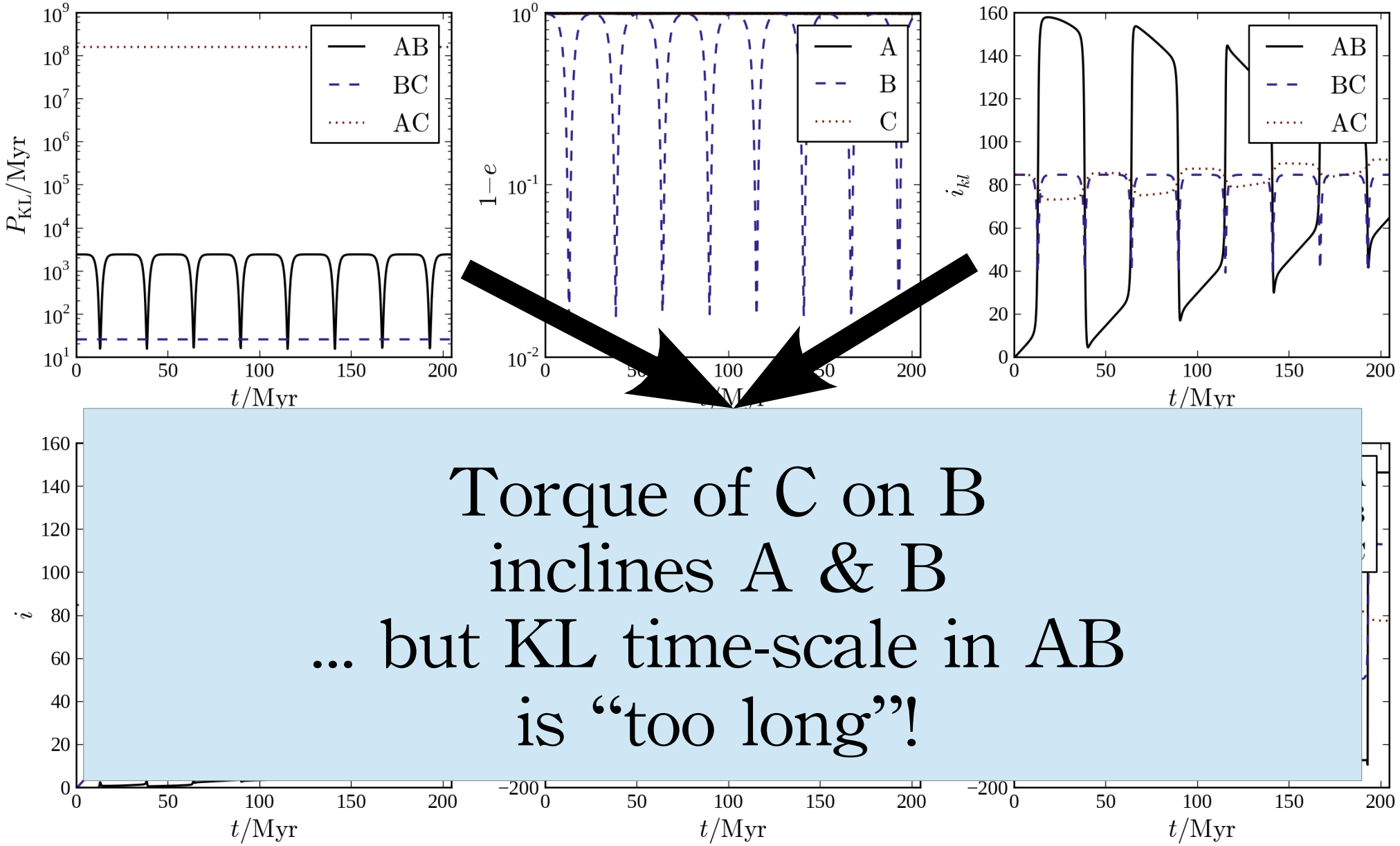






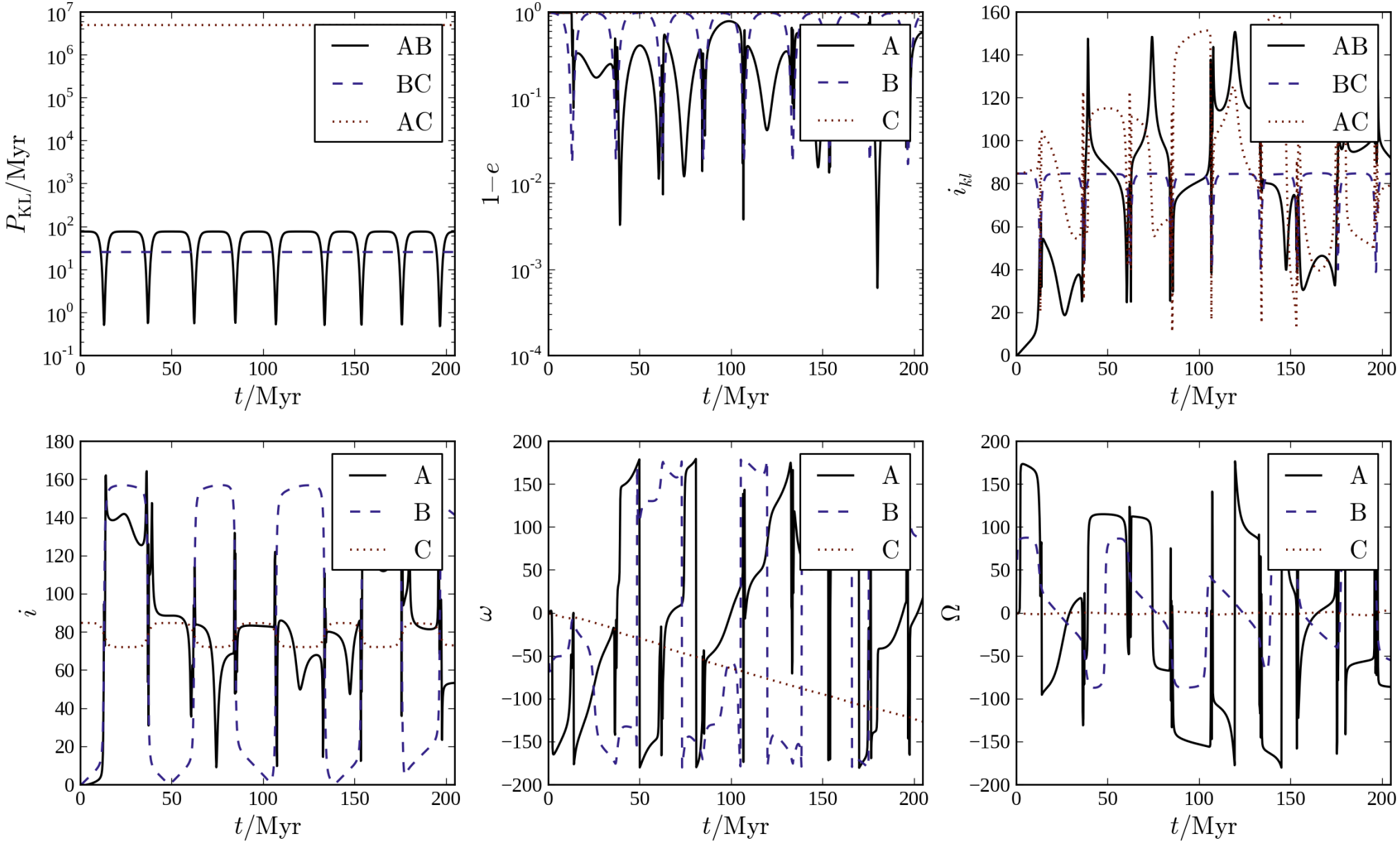


$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 1 \times 10^2$$

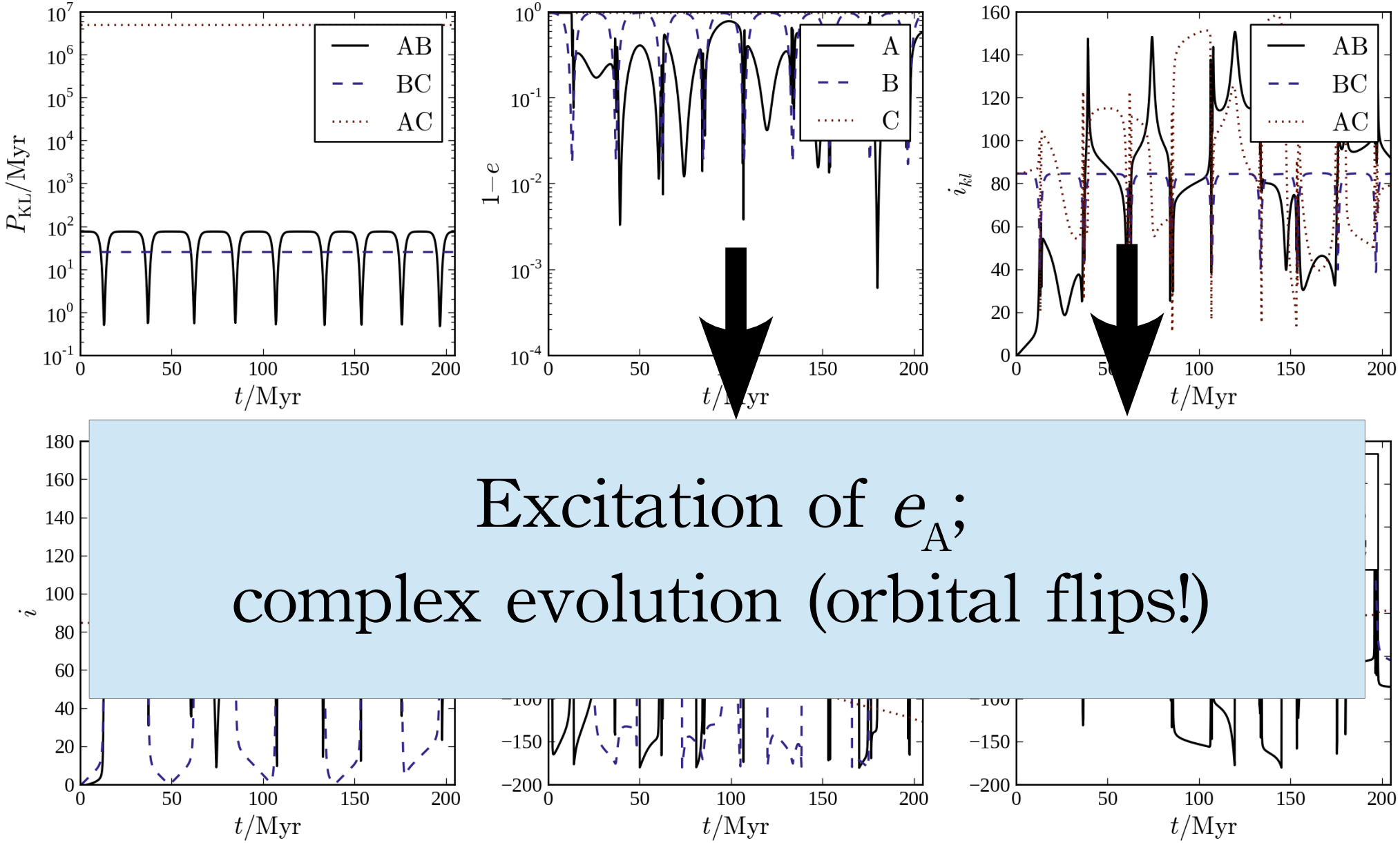


Torque of C on B
inclines A & B
... but KL time-scale in AB
is "too long"!

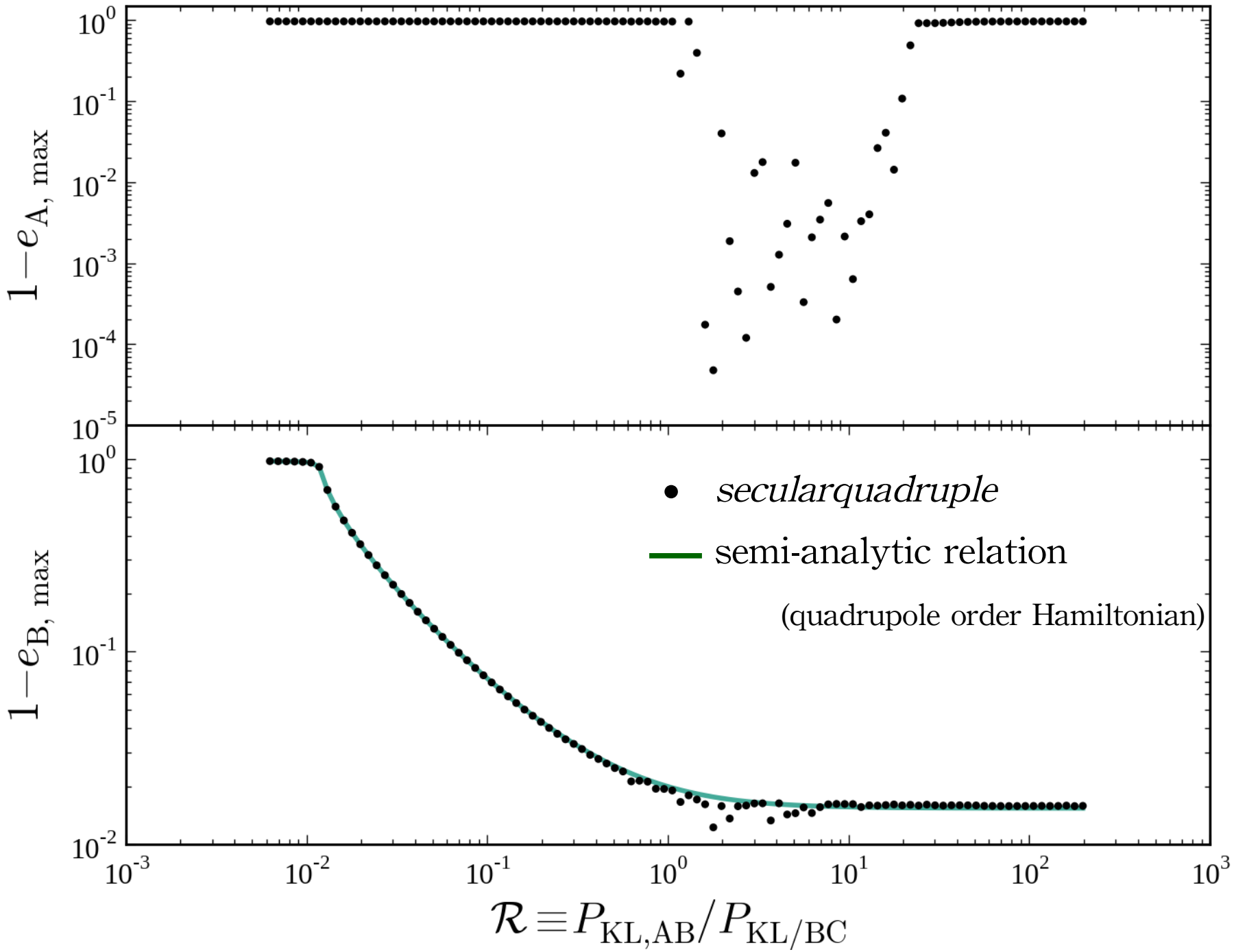
$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 3$$



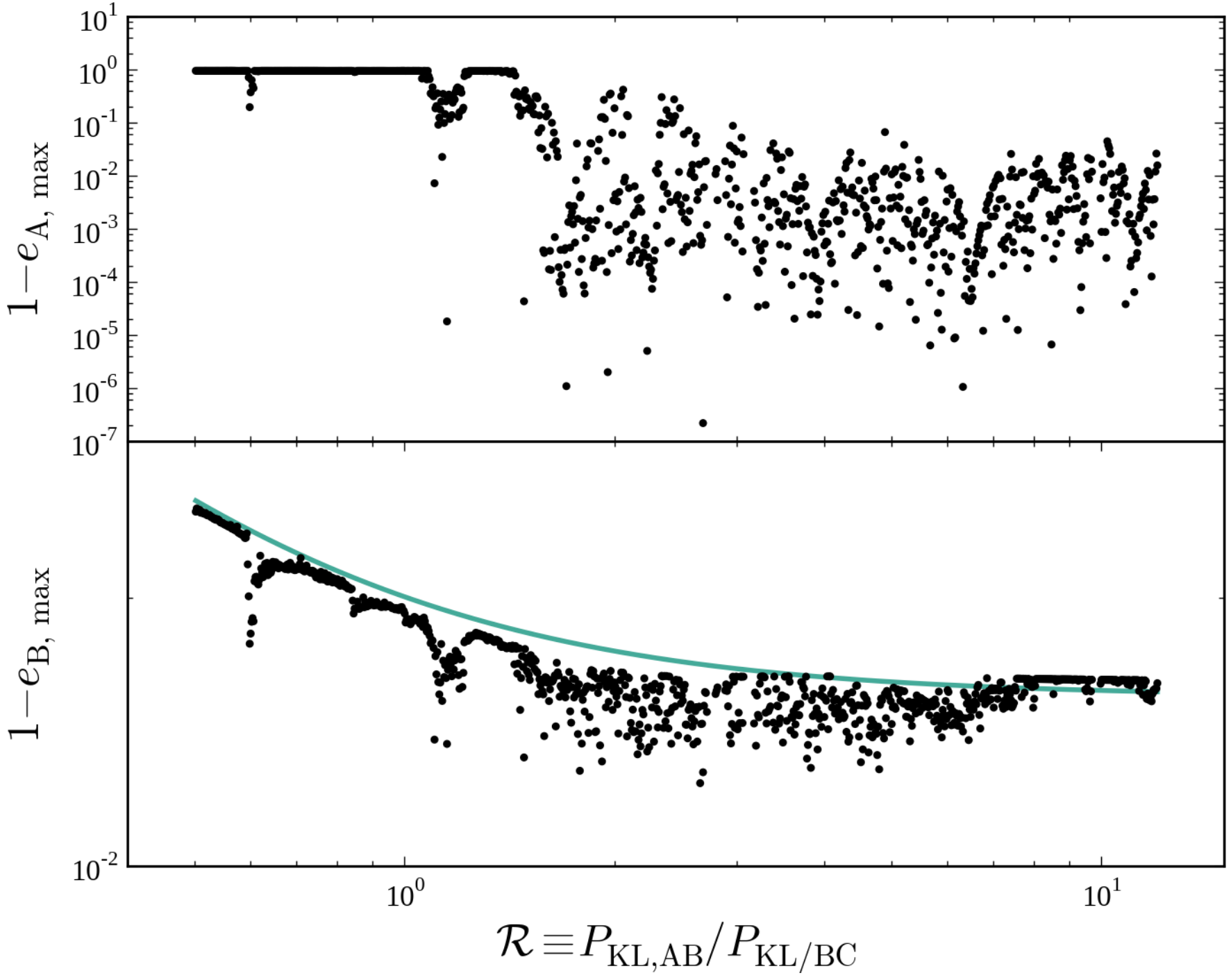
$$\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC} \approx 3$$



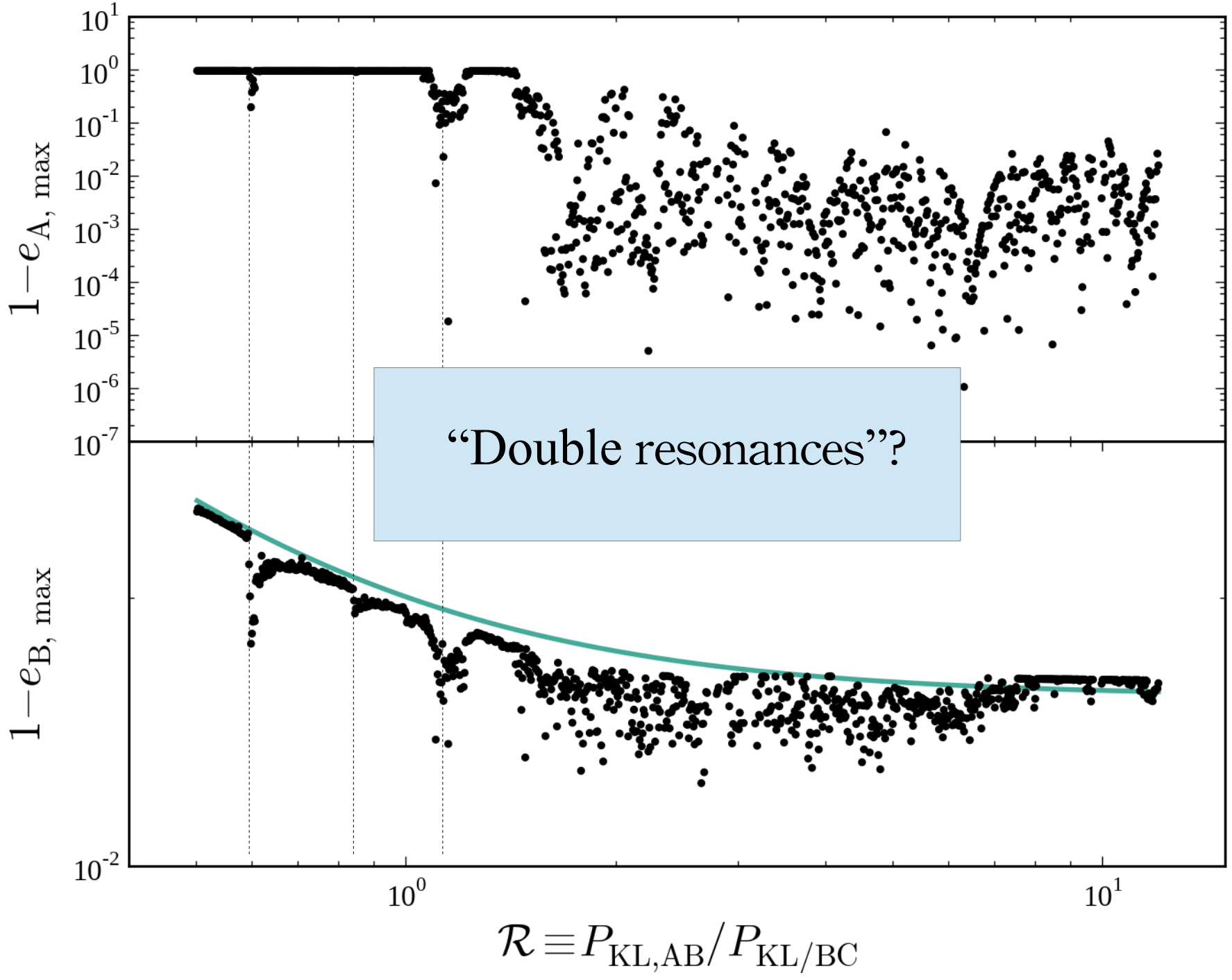
Excitation of $e_{\rm A}$;
complex evolution (orbital flips!)


10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$1-e_{\mathrm{A,\,max}}$
10^{0}
10^{-1}
10^{-2}
$1-e_{\mathrm{B,\,max}}$
secularquadruple
semi-analytic relation
(quadrupole order Hamiltonian)
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
$\mathcal{R} \equiv P_{\mathrm{KL,AB}}/P_{\mathrm{KL/BC}}$

Zoom-in


10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$1-e_{\mathrm{A,\,max}}$
10^{-2}
$1-e_{\mathrm{B,\,max}}$
10^{0}
10^{1}
$\mathcal{R} \equiv P_{\mathrm{KL,AB}}/P_{\mathrm{KL/BC}}$

Zoom-in


10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$1-e_{\mathrm{A,\,max}}$
"Double resonances"?
$1-e_{\mathrm{B,\,max}}$
10^{-2}
10^{0}
10^{1}
$\mathcal{R} \equiv P_{\mathrm{KL,AB}}/P_{\mathrm{KL/BC}}$

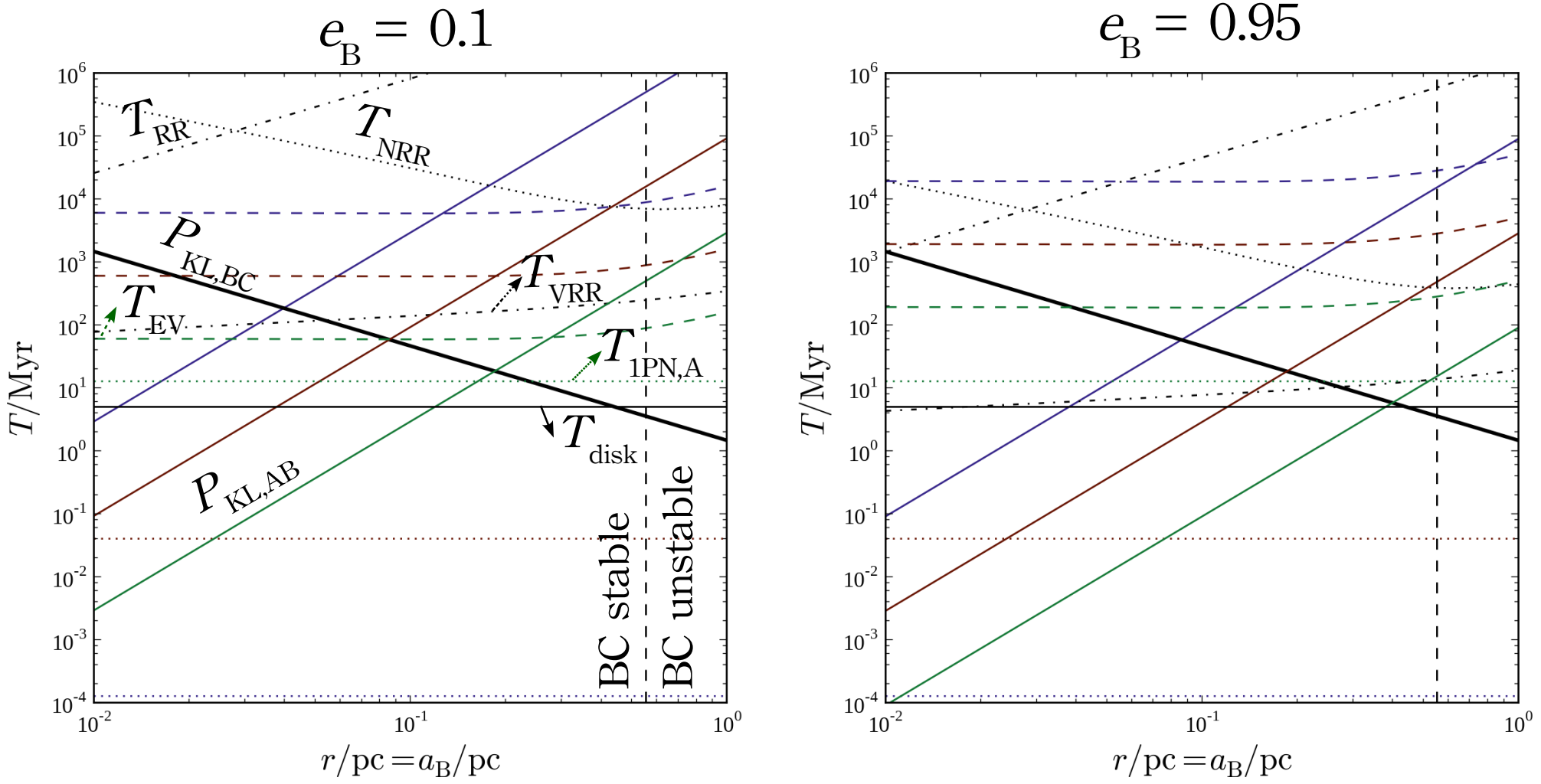
5. Application: binaries in the GC

Binaries orbiting a SBH

(e.g. Hopman 2009, Antonini+ 2010, 2012, Prodan+ 2014)

- tidal force of SBH → disruption (→ S-stars?)
- evaporation
- high e KL cycles due to torque of SBH
  - tidal friction, GWs (LIGO), physical collisions (→ exotic objects?)
  - additional precession (e.g. GR) → quenching
  - effect of torque in addition to SBH?
    - IBH
    - $N^{1/2}$-asymmetry of stellar cluster ("VRR")
    - Large-scale structure, e.g. circumnuclear disk (CND)

Time-scales



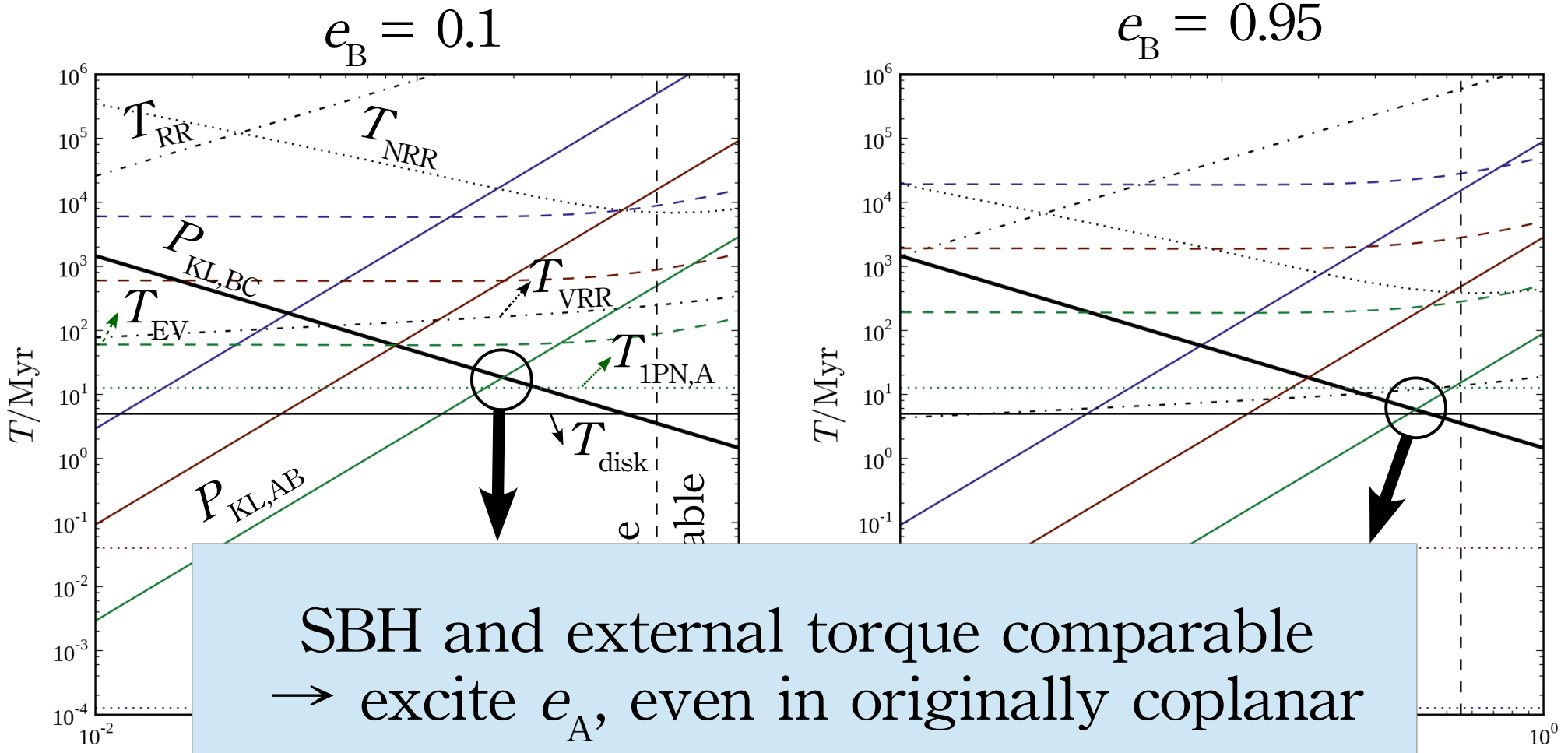
"Fourth body": CND ($m_4 = 10^6$ $M_\odot$) with $a_C = 2$ pc (e_C=0.1)

$a_A = 1$ AU

$a_A = 0.1$ AU

$a_A = 0.01$ AU

Time-scales



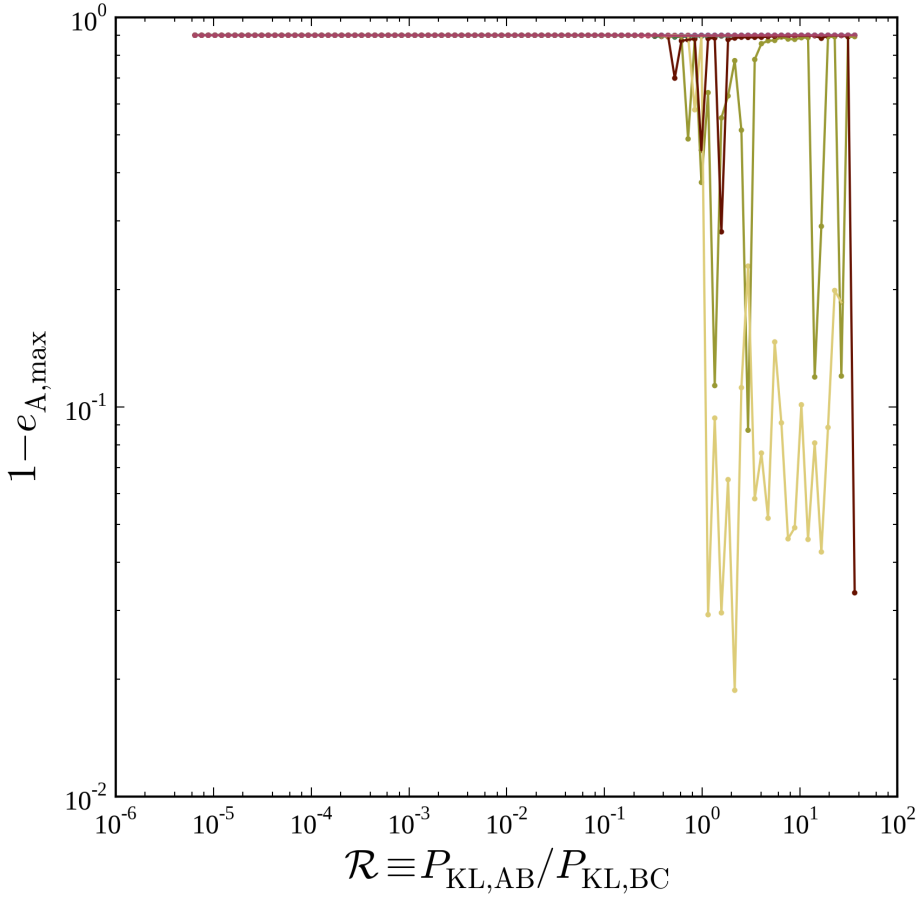
SBH and external torque comparable
→ excite e_A, even in originally coplanar systems ($i_{AB} = 0$)?

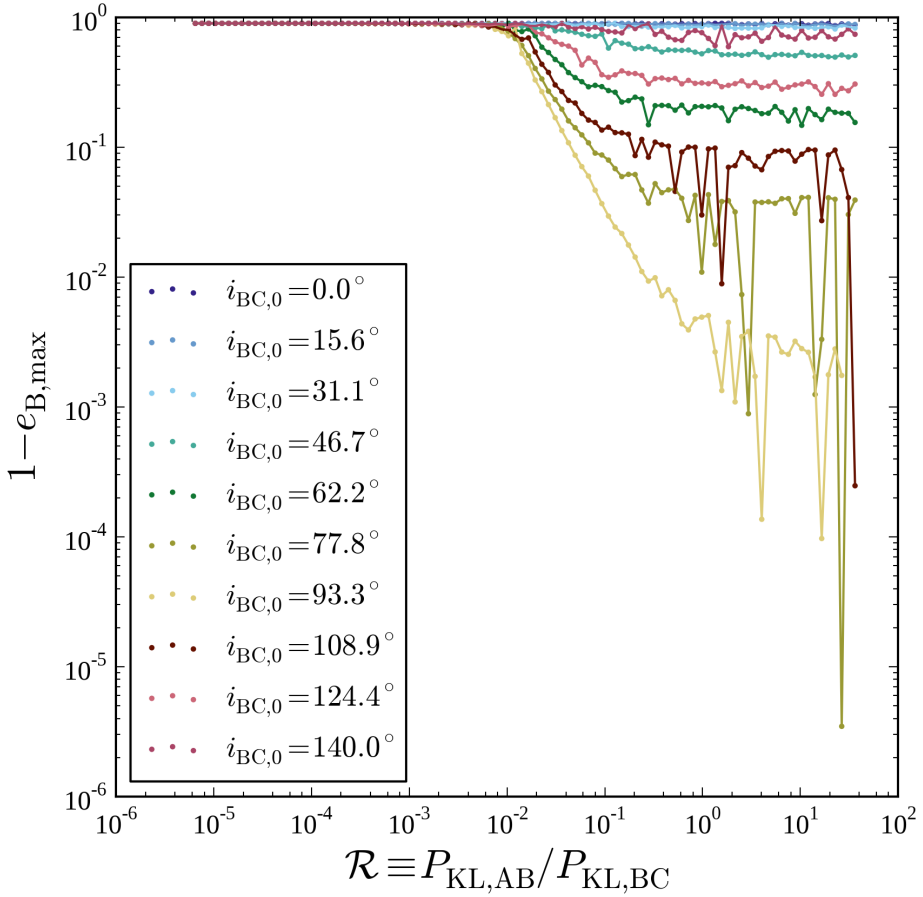
"Fourth body": CND ($m_4 = 10^6\ M_\odot$)
at $a_C =$ 2 pc (e_C=0.1)

a_A = 1 AU
a_A = 0.1 AU
a_A = 0.01 AU

Proof of principle (1PN included)

A & B initially coplanar



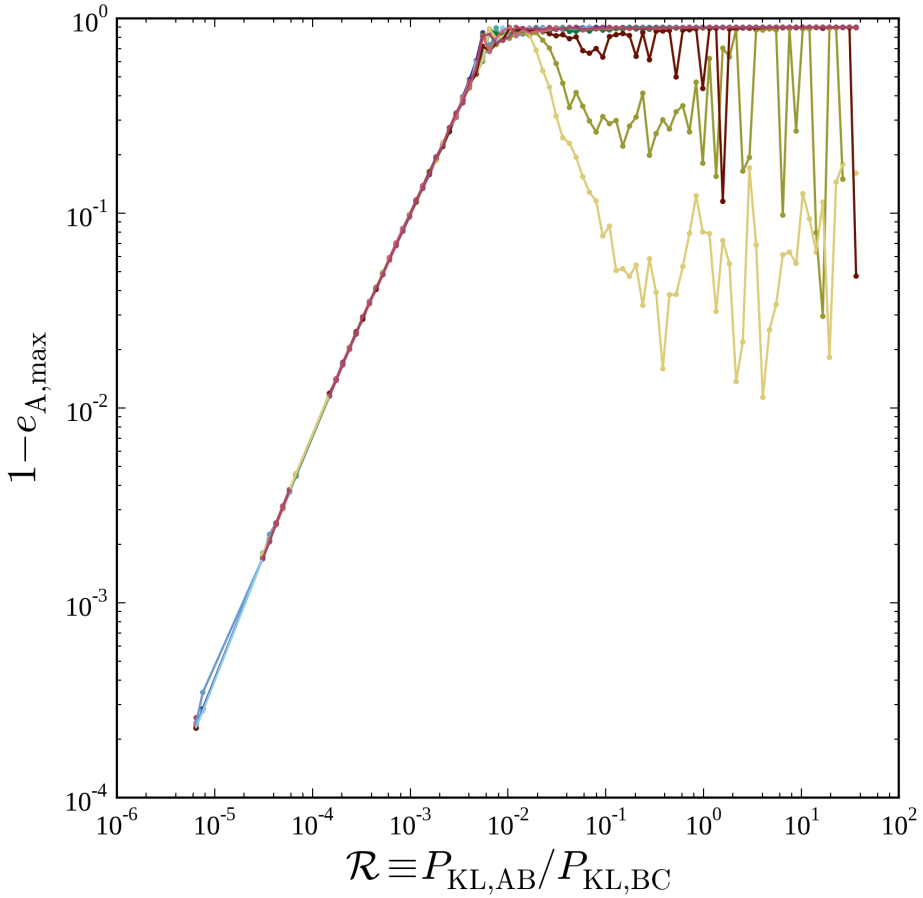


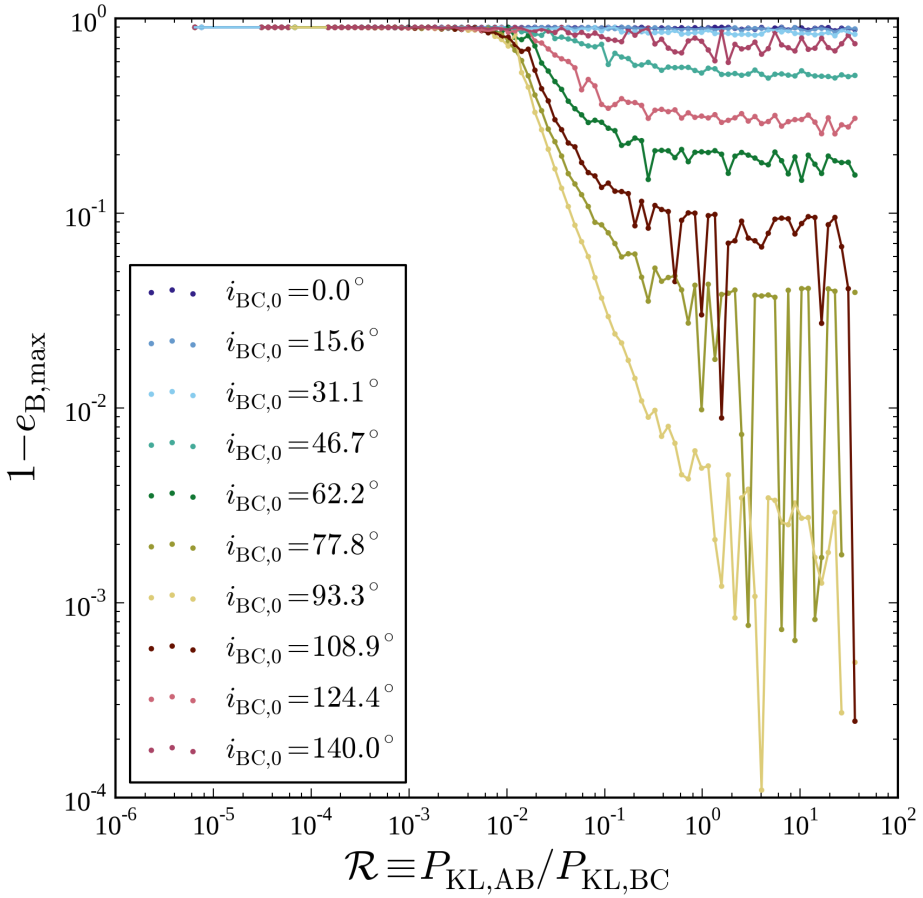
$\mathcal{R} \sim 1 \rightarrow$ KL of C on B (high $i_{\rm BC,0}$) $\rightarrow$ $e_{\rm B}$ excited

$\rightarrow$ $P_{\rm KL,AB}$ reduced $\rightarrow$ $P_{\rm KL,AB} < P_{\rm 1PN,A} \rightarrow$ $e_{\rm A}$ excited!

Proof of principle (1PN included)

A & B initially inclined (90°)





Small $\mathcal{R}\rightarrow$ quenching due to 1PN precession in A
Larger $\mathcal{R}\rightarrow$ quenching again overcome due to high $e_{\rm B}$

6. Summary/conclusions

Different regimes in quadruples broadly described in terms of $\mathcal{R} \equiv P_{\rm KL,AB}/P_{\rm KL,BC}$

- $\mathcal{R} \ll 1$: torque of B on A dominates
  - if A & B coplanar $\rightarrow$ remain coplanar
  - induced precession on B from A quenches KL cycles in B
- $\mathcal{R} \gg 1$: torque of C on B dominates
  - if A & B coplanar $\rightarrow$ induced inclination
  - $P_{\rm KL,AB}$ too long for excitation of $e_{\rm A}$
- $\mathcal{R} \sim 1$: torques comparable
  - induced excitation of $e_{\rm A}$
  - potentially very high $e_{\rm A}$ (orbital flips), even for coplanar A & B
  - $e_{\rm B}$ enhanced ("double resonances"; specific values of $\mathcal{R}$)

- Future work: applications in / implications for
  - binaries in galactic nuclei
  - planetary systems