MHD instabilities

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Introduction

- Magnetic fields are ubiquitous in cosmological objects
- General questions exist about why are they of the form that they are:
  - Why does this particular form remain? STABILITY
  - How does it get to this form? Did this form evolve from some other form? INSTABILITY
- Anything that re-organises magnetic field is a transport phenomenon therefore instabilities are of great interest
- Have discussed purely hydrodynamic instabilities. Now discuss the role of magnetic field in the instability process.
- Basically, two possibilities:
  - Existing hydro instability affected by presence of magnetic field
  - Instability driven by presence of magnetic fields

There exists a massive catalogue of MHD instabilities with many from many different communities …

Some from our friends in the plasma community …

List of plasma instabilities

- Bennett pinch instability (also called the z-pinch instability)
- Beam acoustic instability
- Bump-in-tail instability
- Buneman instability,[2] (same as Farley-Buneman instability?)
- Cherenkov instability,[3]
- Chute instability
- Coalescence instability,[4]
- Collapse instability
- Counter-streaming instability
- Cyclotron instabilities, including:
  - Alfven cyclotron instability
  - Electron cyclotron instability
  - Electrostatic ion cyclotron Instability
  - Ion cyclotron instability
  - Magnetoacoustic cyclotron instability
  - Proton cyclotron instability
  - Nonresonant Beam-Type cyclotron instability
  - Relativistic ion cyclotron instability
  - Whistler cyclotron instability
- Diocotron instability,[5] (similar to the Kelvin-Helmholtz fluid instability).
- Disruptive instability (in tokamaks)
- Double emission instability
- Drift wave instability
- Edge-localized modes [2]
- Electrothermal instability
- Farley-Buneman instability
- Fan instability
- Filamentation instability
- Firehose instability (also called Hose instability)
- Flute instability
- Free electron maser instability
- Gyrotron instability
- Helical instability (helix instability)
- Helical kink instability
- Hose instability (also called Firehose instability)
- Interchange instability
- Ion beam instability
- Kink instability
- Lower hybrid (drift) instability (in the Critical ionization velocity mechanism)
- Magnetic drift instability
- Modulation instability
- Non-Abelian instability (see also Chromo-Weibel instability)
- Chromo–Weibel instability
- Non-linear coalescence instability
- Oscillating two stream instability, see two stream instability
- Pair instability
- Parker instability (magnetic buoyancy instability)
- Peratt instability (stacked toroids)
- Pinch instability
- Sausage instability
- Slow Drift Instability
- Tearing mode instability
- Two stream instability
- Weak beam instability
- Weibel instability
- z-pinch instability, also called Bennett pinch instability

Introduction (cont)

- Clearly impossible to describe all instabilties in this lecture :)
- Attempt to describe some that are
  - relevant to astrophysics
  - BUT common to many communities
  - AND simple :)
- Cover mainly
  - the idea of where they come from intuitively (rather than rigourously derive the linear stability analysis)
  - constraints on the form they may take

Basic methodology

- The basic methodology of examining instabilities in MHD is exactly the same as for HD:
  - Take the equations
  - Linearise about an equilibrium solution
  - Add some perturbations and see what happens (mathematically)
  - (a) Look for NORMAL MODE solutions $e^{i\omega t} e^{i\,\mathbf{k}\cdot\mathbf{x}}$ and find a relationship between the growth rate ω and the wavenumbers of the disturbance $\boldsymbol{k}$ and the parameters of the problem. $\omega^2<0$ => positive growth rate and instability.
  - (b) Use an ENERGY VARIATIONAL approach: calculate δW, the change in potential energy associated with the disturbance, and look for disturbances with $\delta W < 0$ (the existence of any one means unstable; for stable, need $\delta W > 0$ for ALL types of disturbances [harder])

The compressible MHD equations

Cons of mass

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

Cons of mom

Mmm … nonlinearity!

$$\partial_t \rho \mathbf{u} + \nabla \cdot (\underbrace{\rho \mathbf{u}\mathbf{u}}_{\text{advection}} - \underbrace{\alpha \mathbf{B}\mathbf{B}}_{\text{tension}}) = \overbrace{-\nabla p_T}^{\text{pressure}} + \underbrace{\rho \mathbf{g}}_{\text{buoyancy}} - \overbrace{\sigma C_k Ta^{1/2} (\mathbf{\Omega} \times \rho \mathbf{u})}^{\text{rotation}}$$

$$+ \underbrace{\sigma C_k (\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}))}_{\text{diffusion}} + \overbrace{\mathbf{F}}^{\text{forcing}}$$

Cons of energy

Mmm … nonlinearity!

$$\partial_t T + \underbrace{\nabla \cdot (\mathbf{u} T)}_{\text{Advection}} + \underbrace{(\gamma - 2) T \nabla \cdot \mathbf{u}}_{\text{Compression}} = \underbrace{\frac{\gamma C_k}{\rho} \nabla \cdot (\frac{\kappa(z)}{\kappa(0)} \nabla T)}_{\text{diffusion}} + \underbrace{H_\eta + H_\upsilon}_{\text{Ohmic \& viscous heating}}$$

Induction

Mmm … nonlinearity!

$$\partial_t \mathbf{B} = \underbrace{\nabla \times (\mathbf{u} \times \mathbf{B})}_{\text{Induction}} + \underbrace{C_k \zeta \nabla^2 \mathbf{B}}_{\text{diffusion}}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$p_T = p_g + p_m = \rho T + \underbrace{\frac{\alpha |\mathbf{B}|^2}{2}}_{\text{magnetic pressure}}$$

$$\alpha = \sigma \zeta C_k{}^2 Q$$

Compressible MHD equations

(Navier-Stokes + induction)

Magnetohydrostatics

- For true instability, need to perturb an equilibrium.
- Quick thoughts: ideal (no diffusion), static => all that is left is

$$0 = -\nabla p + \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}$$

  - If the magnetic field is "weak" (magnetic pressure << gas pressure; plasma β>>1): Lorentz small, small magnetic adjustments only to hydrostatic balance
  - If the magnetic field is "strong" (plasma β>>1): Lorentz dominates

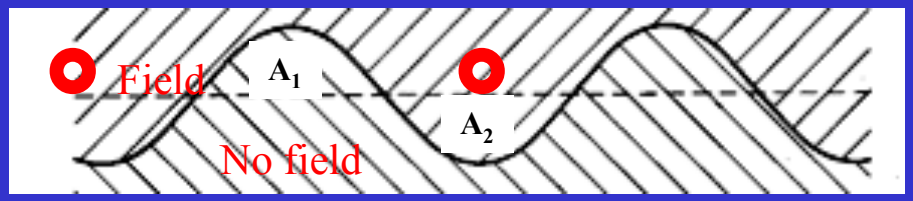
$$0 = \mathbf{J} \times \mathbf{B}$$

  - "Force-free" . Possibilities: J ~ B, J=0 ("current-free" or "potential field" $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0 \Rightarrow \nabla^2 \mathbf{B} = 0$)
  - Many real situations have regions of both types e.g. stars: interior pressure balanced, atmosphere force-free (ish)
- Often HARD to construct magnetic equilibria

Interchanges

- A whole class of (ideal) instabilities, with many common examples
- Basically involves swapping of magnetic field lines without (further) bending
- If magnetic energy decreases in swap, then instability proceeds

Imagine an interface between two incompressible plasmas (or easiest to think of, between upper layer=plasma and lower layer=no plasma):



- Perturbation up, area A_1
- Perturbation down, area A_2
- Initial (upper) field B_1; final (upper) field B_2. Effectively transferring field from A_1 to A_2
- Flux conserved: $B_1A_1=B_2A_2$ or $B_2=B_1A_1/A_2$
- Energy difference between same volumes:

$$\frac{B_1^2}{2\mu}V-\frac{B_2^2}{2\mu}V=\frac{B_1^2}{2\mu}V\left(\frac{A_1^2}{A_2^2}-1\right)$$

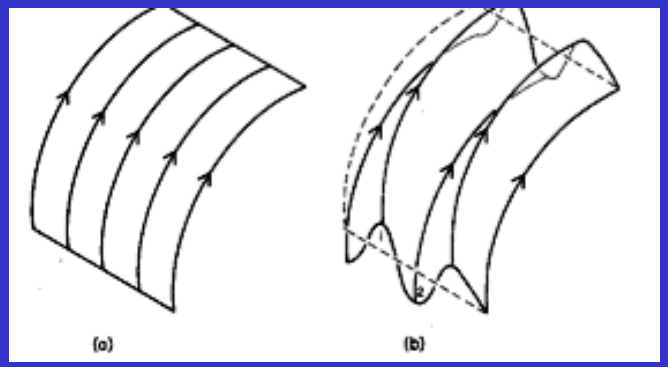
- Negative (lose energy) and instability if $A_1 < A_2$

→ distinguish between up perturbation and down and their areas

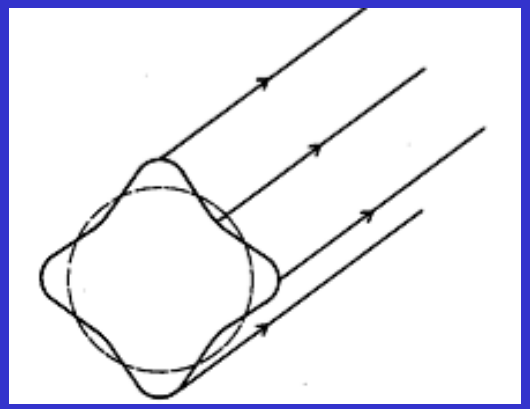
→ Can only preserve volume for different areas if length changes

Canonical Interchange: Fluting

- Canonical example: Concave field containing a field-free plasma



or



- Since outward perturbation must lengthen field lines, is unstable
- Can think of it as:

  ⇒ In moving field from upper area to lower area, area increases so field must decrease to preserve flux.

  ⇒ Decreasing field means decreasing energy and instability

- Normal mode analysis: $e^{i\omega t} \rightarrow \omega^2 \sim -\frac{k}{R_C}$
- High k (short waves) and small radius of curvature grow fastest
- Can suppress interchanges by twisting field making it hard to rise

Interchange: Rayleigh-Taylor

- MHD version of classical hydro instability: dense fluid over less dense fluid is gravitationally unstable

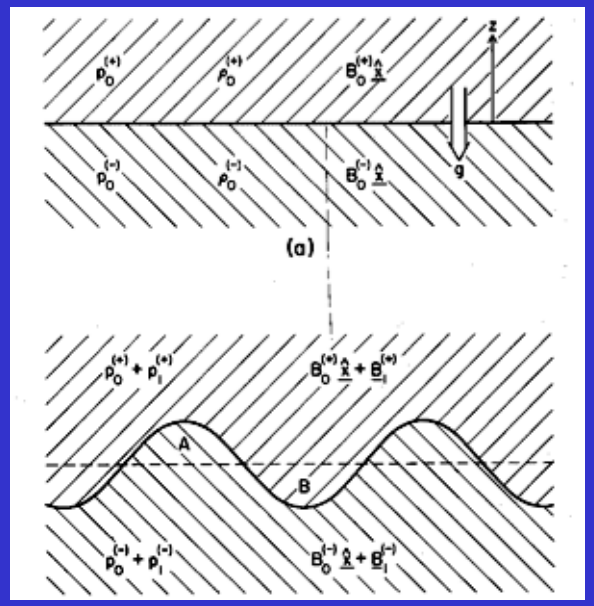
$$\omega^2 = -gk\frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}}.$$

Any density difference will do. Is an interchange in density: swap dense upper for light lower => dW<0

- Now dense plasma layer overlaying a less dense plasma layer:
- Vertical field $B_0\underline{z}$ only:

Modifies growth rate slightly for short waves (long = same)

$$i\omega \approx \frac{g\sqrt{\mu}}{B_0}\left(\sqrt{\rho_0^{(+)}} - \sqrt{\rho_0^{(-)}}\right),$$



- Horizontal field $\mathbf{B_0} = (B_0,0,0)$:

$$\omega^2 = -gk\frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}} + \frac{2B_0^2k_x^2}{\mu(\rho_0^{(+)} + \rho_0^{(-)})},$$

$k_x=0$: field perp to pert wave; interchange; dispersion reln unaffected; MOST UNSTABLE MODE

$k=k_x$: Field parallel; undular modes; growth modified; instability only if $k<k_C$

$$k_c = \frac{(\rho_0^{(+)} - \rho_0^{(-)})g\mu}{2B_0^2}.$$

Tension opposes growth of instability.

Don't want to bend => long waves

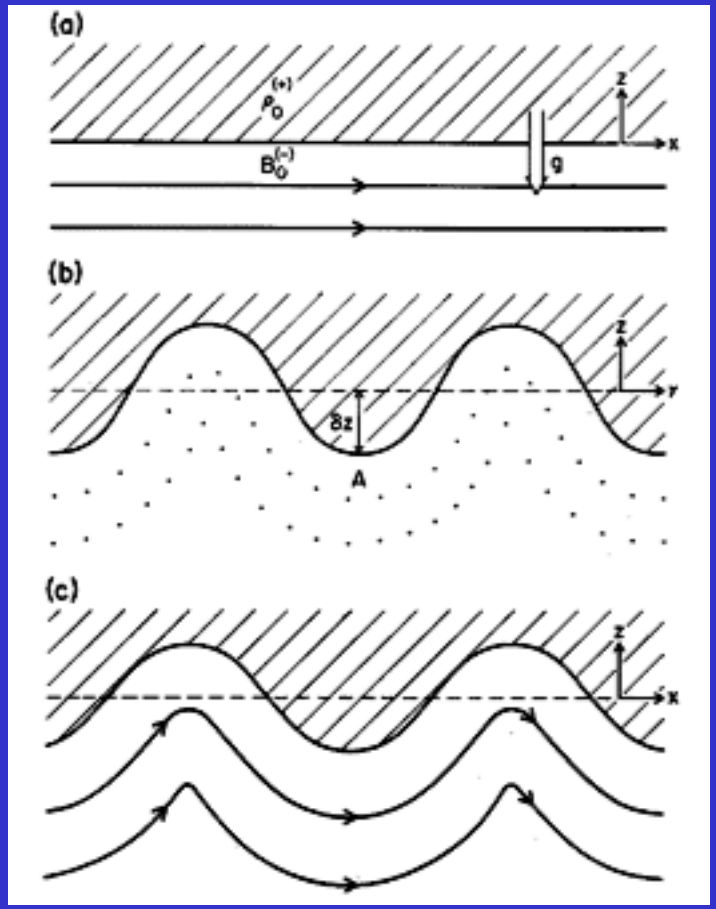
Interchange: Rayleigh-Taylor

- Special case: Kruskal Schwarzchild (1954)
- Plasma over evacuated magnetic field $\rho^{(-)}=0$
- Dispersion relation:

$$\omega^2 = -gk + \frac{k_x^2 B_0^{(-)2}}{\mu \rho_0^{(+)}}$$

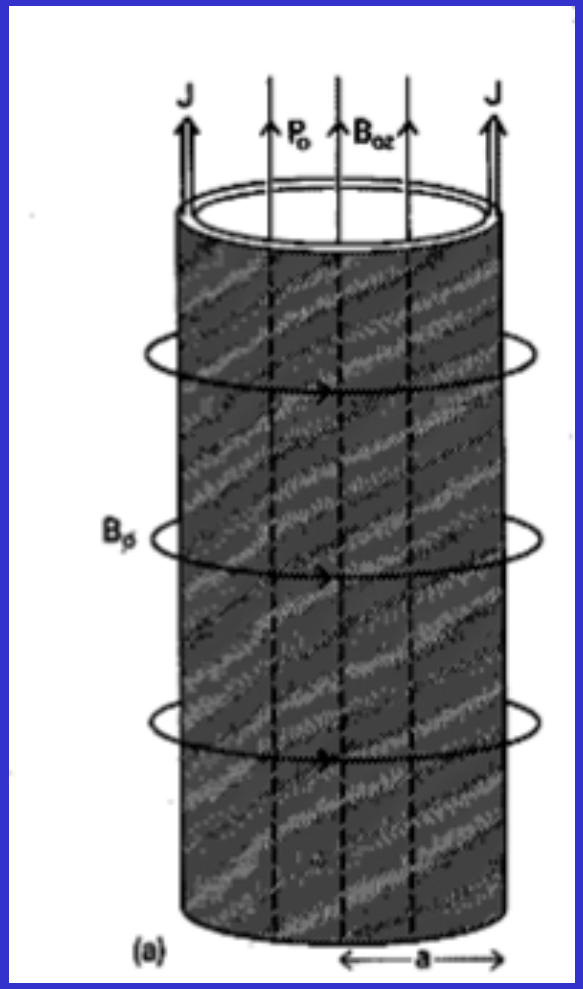
- Most unstable

$$i\omega = (gk)^{1/2}.$$



Other interchanges: A common tale of complicated MHD instabilities

- From the plasma side of things:
  - Thermonuclear fusion program
  - Heat gas hotter than centre of the sun
  - Confine plasma long enough for nuclear reactions to take place
  - Hope produce more energy than put it!
  - Instabilities of plasma a major obstacle
- "Linear pinch" -- cylindrical confinement device
- Column of plasma confined by magnetic field
- Useful to us since similar to
  - Interior of stars
  - Magnetic flux tubes



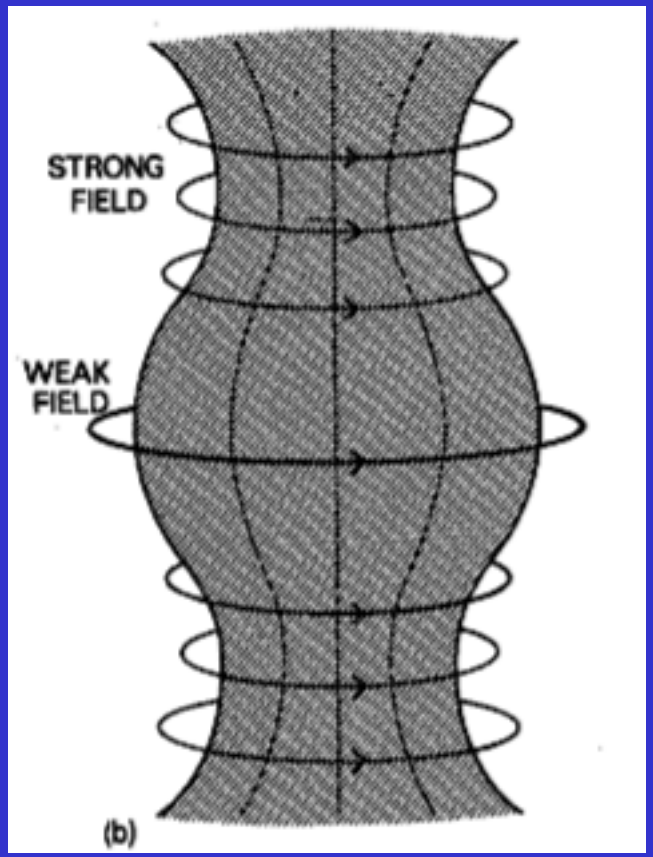
Tayler (1957): m=0 mode = "sausage"; m=1 "kink"; m=2,3, … general interchanges

Sausage instability

- No B_z; purely toroidal field
- Initial equilibrium with radially inward Lorentz force balanced by outward pressure gradient
- Unstable to interchange due to curvature as before (but different alignment)
- Dispersion relation:

$$\omega^2 = -2\left(\frac{p_0}{\rho_0}\right)\frac{k}{R_c^2}$$

- Mechanism:
  - Shrink rings
  - B increases
  - JxB increases
  - Radial inward force increases
  - Nothing to counter => instability



Sausage stabilisation

- Add B_z to interior plasma; stabilises sausage instability
- Magnetic pressure of Bz "pushes back to oppose squeezing"
- Pressure balance at interface:

$$p_0 + \frac{B_{0z}^2}{2\mu} = \frac{B_\phi^2}{2\mu},$$

- Dispersion relation:

$$\omega^2 = -\frac{2p_0}{\rho_0 a^2} + \frac{B_{0z}^2}{\mu\rho_0 a^2}.$$

- Leads to condition for stability:

$$B_{0z}^2 > \frac{1}{2}B_\phi^2$$

Kink instability

- Configuration is still unstable to the KINK INSTABILITY though!
- Purely azimuthal or with B_z is unstable to a kinked perturbation:

$$\xi = \xi(R)\exp(i(\phi + kz) + i\omega t)$$

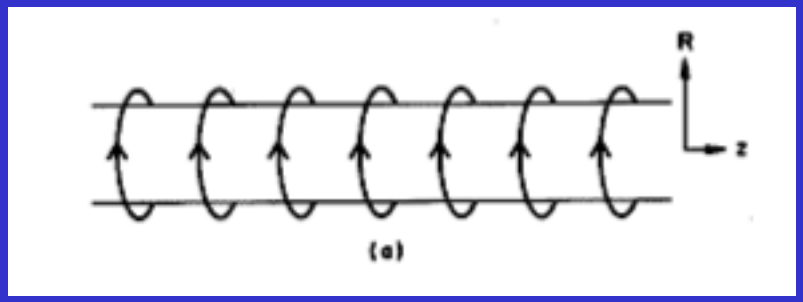
- "m=1" mode retains circular cross-section of tube and perturbation is a kink of the tube into a helix
- Without B_z, unstable for all k
- With B_z, unstable for wavelengths long enough so that the pitch of the perturbation follows the pitch of the helix i.e. the crests/troughs of the perturbations follow the fieldlines of tube

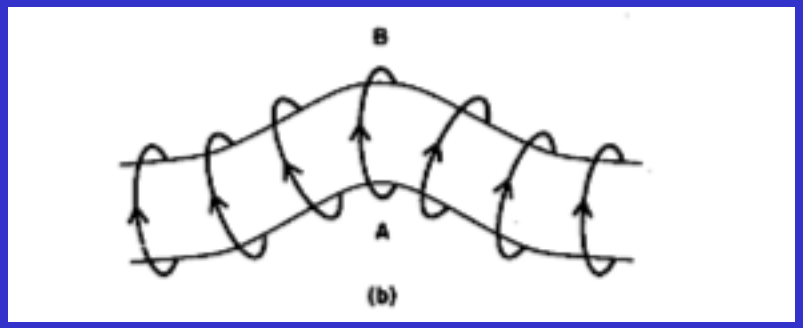
$$B_\phi/R + kB_z \geqslant 0$$

- In terms of twist $\Phi = 2LB_\phi/(RB_z)$, criterion equivalent to $k \geqslant -\frac{\Phi}{2L}$
- For a torus, length 2L, $\Phi \geqslant 2\pi$
- Note: perturbation Lorentz force $(\mathbf{j}_1 \times \mathbf{B}_0)$ is zero
- Note: Two equal and opposite helical kinks = a lateral kink

Lateral kink instability

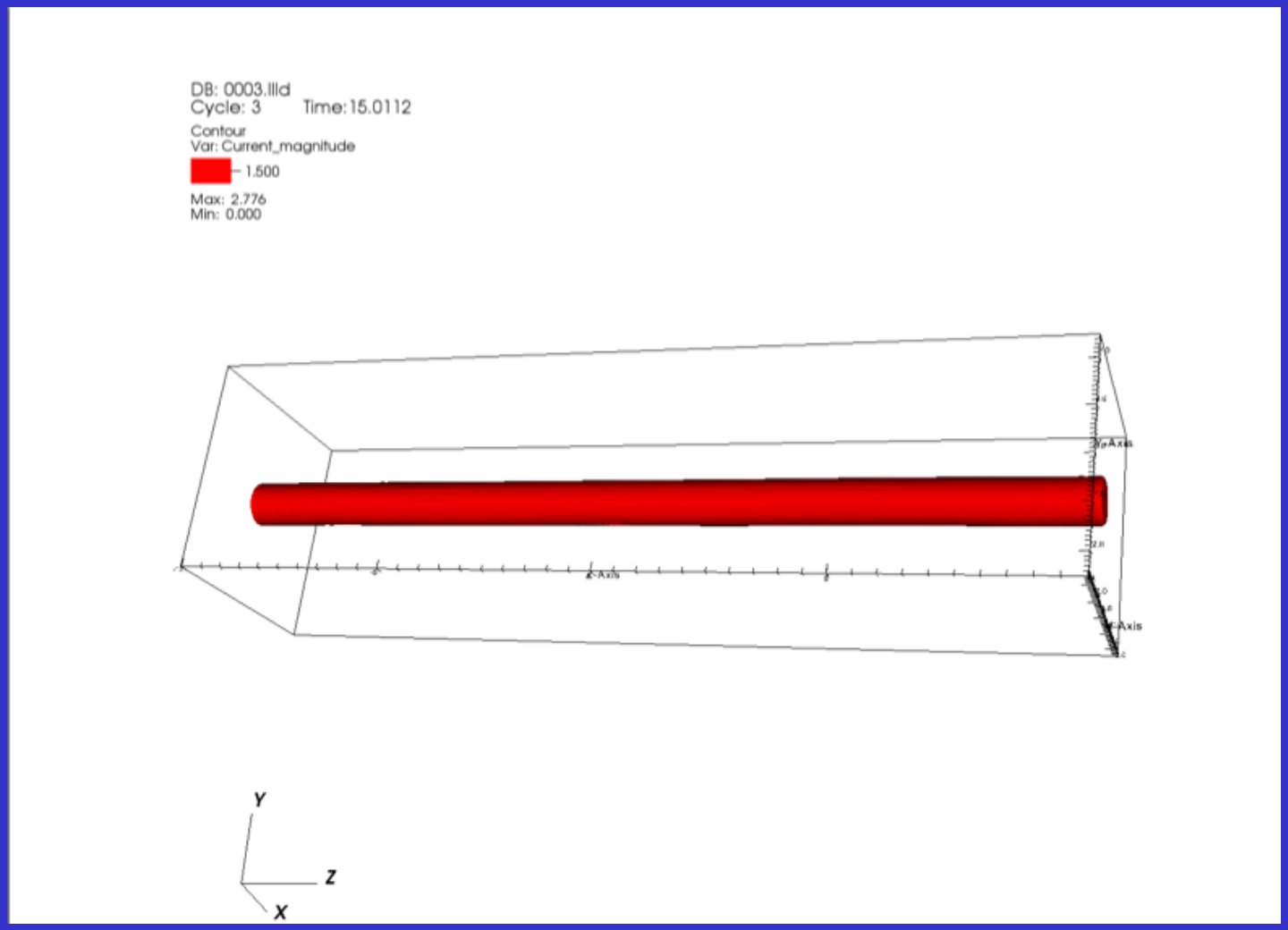
- No helical twist to tube, just bent to the side (all in one plane)





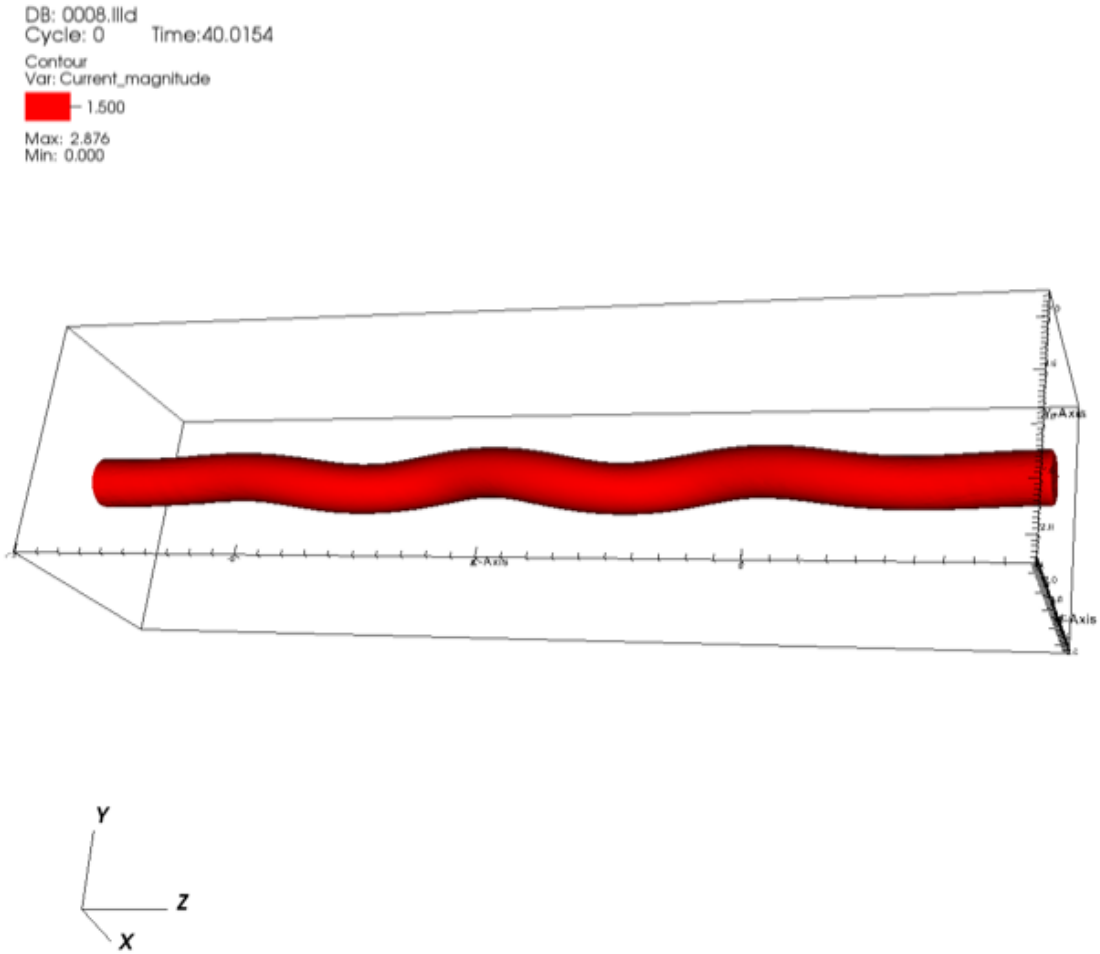
- Can think of as
  - Where perturbed field lines come together, magnetic pressure is increased
  - Where peturbed field lines come apart, magnetic pressure is decreased
  - Magnetic pressures then contiue to drive the perturbation
- Can be stabilised by adding axial magnetic field; adds magnetic tension to resist perturbation

Helical kink instability

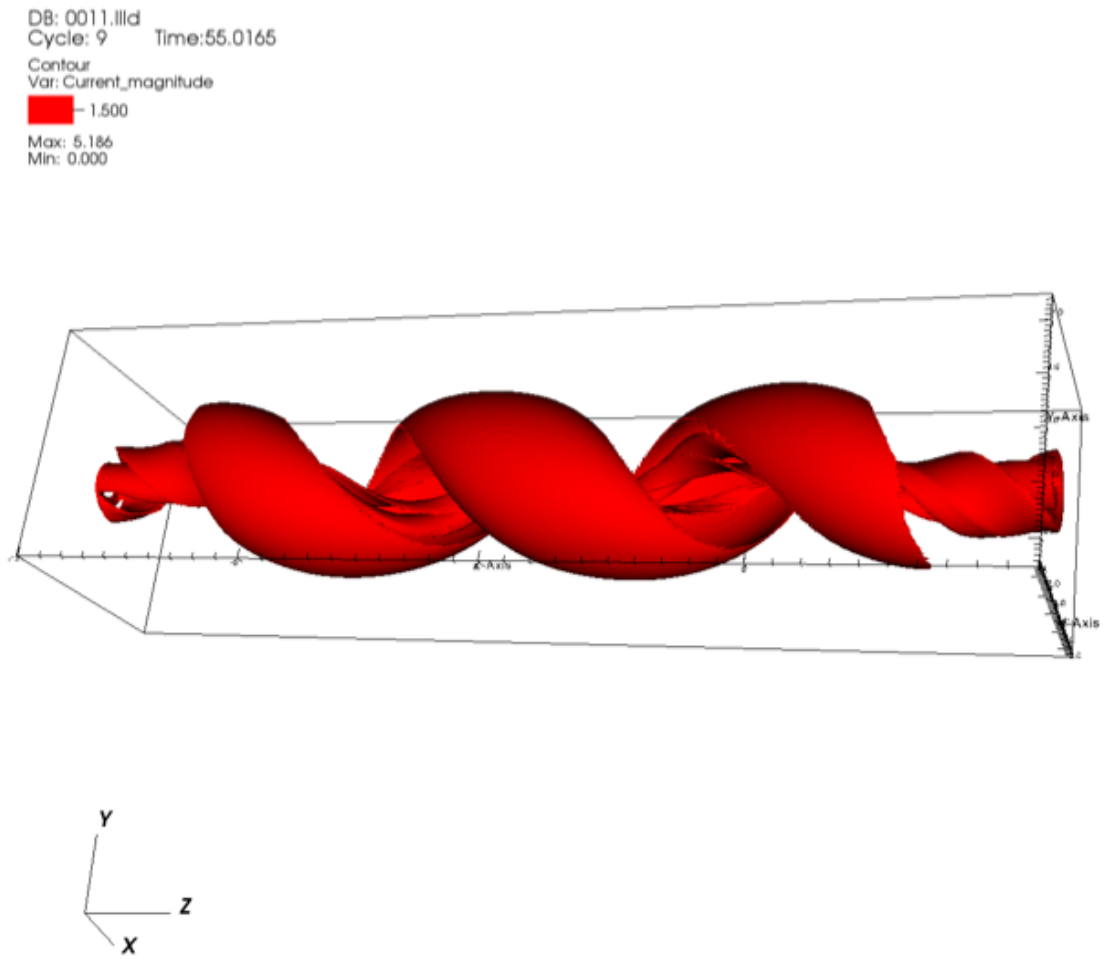


Alan Hood

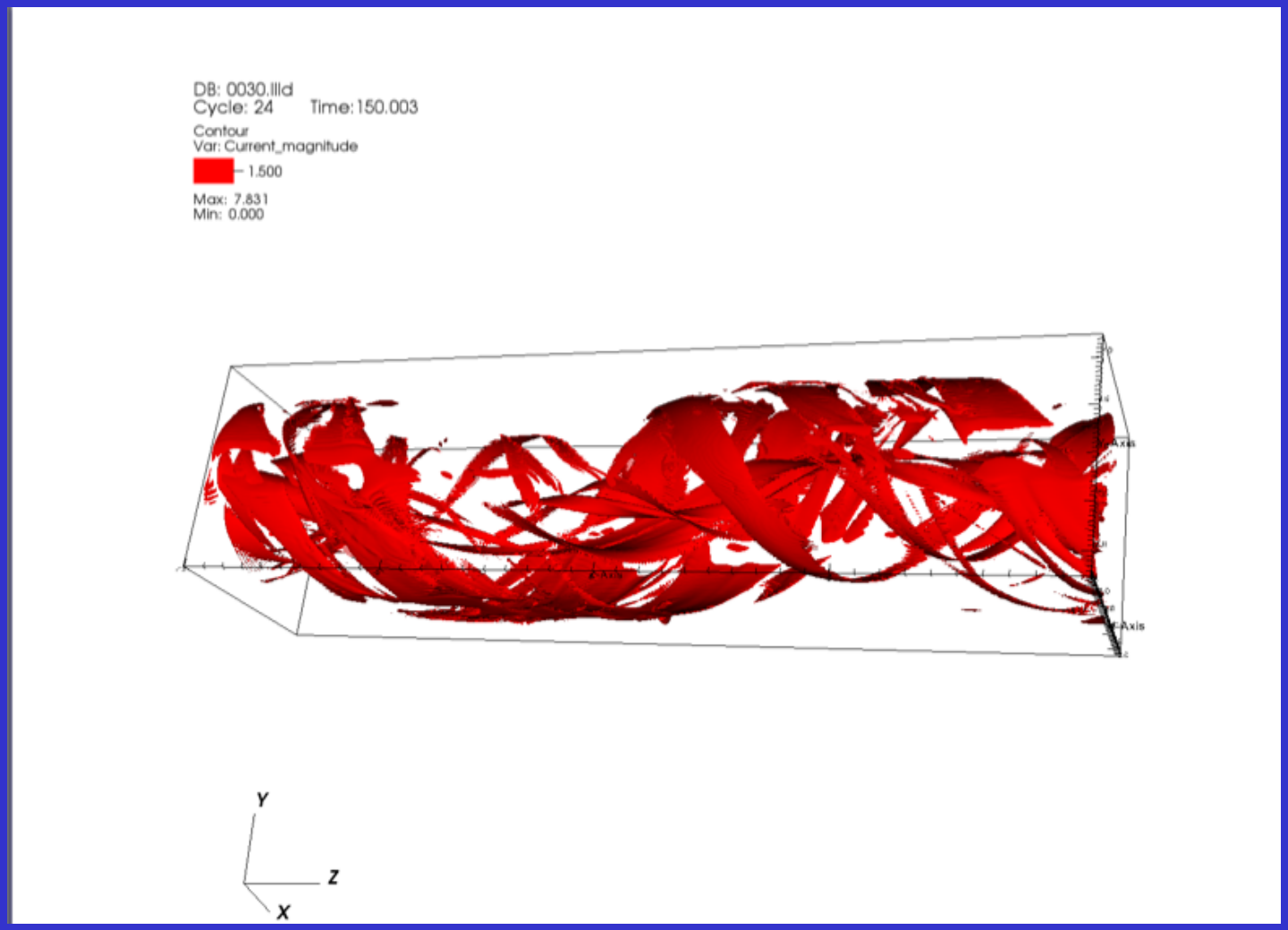
Helical kink instability



Helical kink instability



Helical kink instability

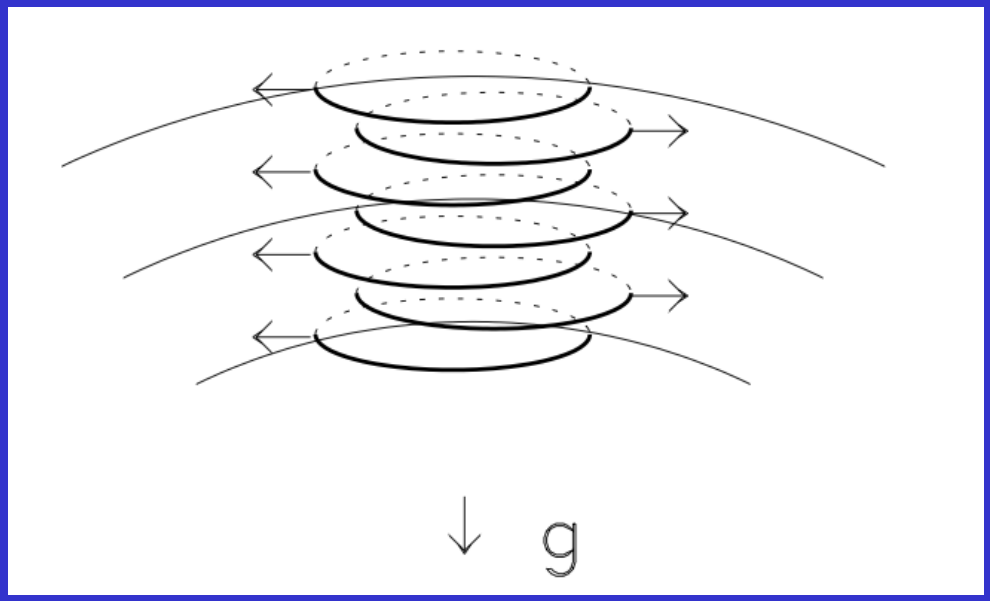


Tayler instability

- Such instabilities can be shown to exist in more complications systems:
  - toroidal pinched discharges
  - compressible
  - with more general forms of B_0 than in original paper
- Furthermore, in a series of papers by Tayler 1973 developed these ideas further to represent the fields in the interiors of stars -- basically, by adding stratification through the presence of gravity, $\mathbf{g}$

Tayler instability

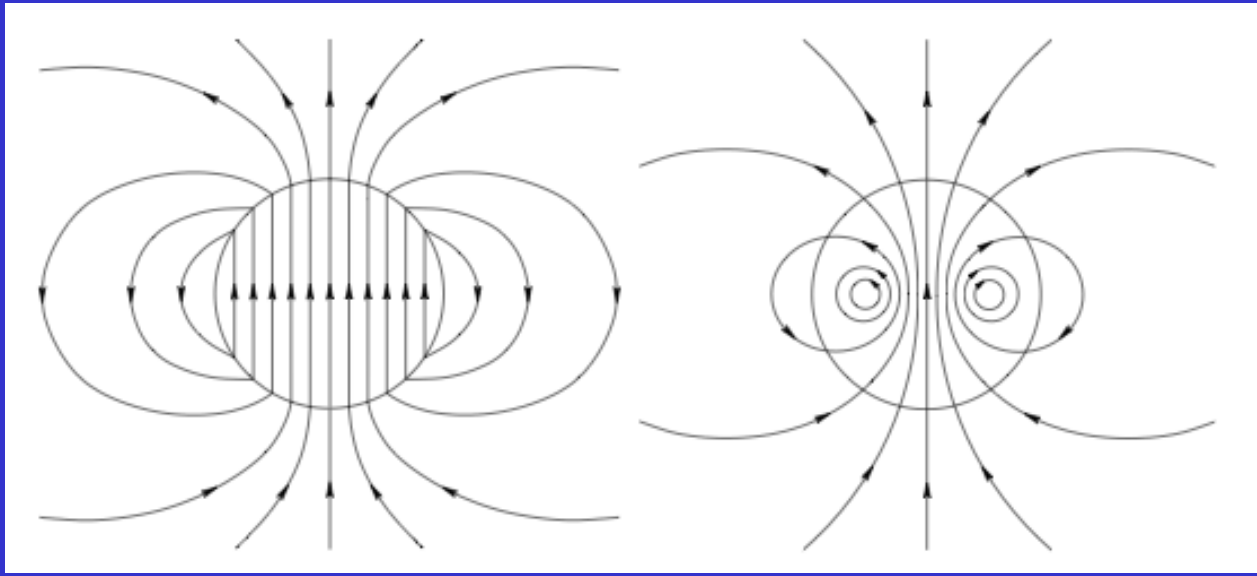
Tayler basically discovered that, for adiabatic perturbations and stable stratification, purely toroidal field is unstable



- Essentially, m=1 lateral kink mode near star's rotational axis
- Works in stratified since m=1 perturbation maintains cross-sectional area and therefore can be ENTIRELY perpendicular to gravity and therefore do no extra work
- Largest growth rates generally O(1/Alfven crossing time of system)

Tayler instability

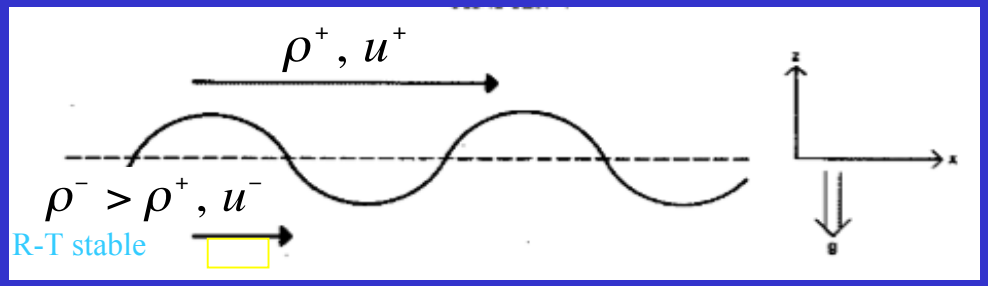
Furthermore, due to Markey and Tayler (1973), Wright (1973), Flowers and Rudermann (1977), purely poloidal axisymmetric field is unstable



– Internal fields of stars can only be stable if they consist of MIXED LINKED POLOIDAL AND TOROIDAL (i.e. twisted field)

Kelvin-Helmholtz

Hydro instability of shear flows:



Unstable for

$$k > \frac{g(\rho^{(-)2} - \rho^{(+)2})}{\rho^{(-)}\rho^{(+)}(U^{(-)} - U^{(+)})^2}$$

… or in the continuously stratified case, stable if

$$\mathrm{Ri} \equiv -\frac{g\, d\rho/dz}{\rho(dU/dz)^2} > \frac{1}{4}$$

(Kinetic energy released not enough to overcome work against gravity)

Always unstable at some wavelength if there exists shear. (Note: inviscid; diffusion may damp out high wavenumber perturbations)

If magnetic fields in layers:

- Fields into paper = interchanges: no change in stability
- Fields B^+ and B^- in x direction:

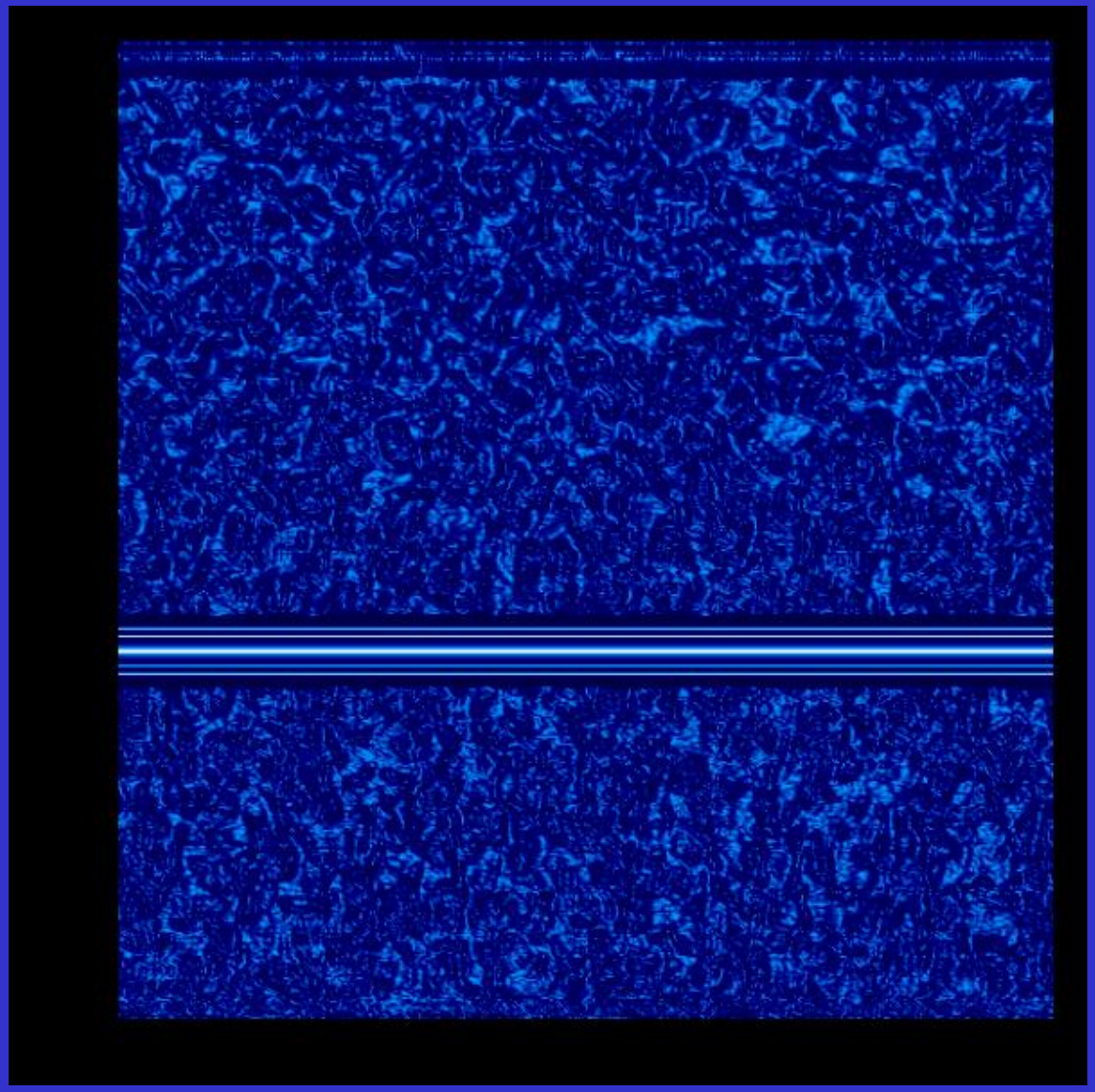
If

$$\frac{B^{(-)2} + B^{(+)2}}{\mu\rho^{(-)}\rho^{(+)}}(\rho^{(-)} + \rho^{(+)}) \geq (U^{(-)} - U^{(+)})^2$$

instability suppressed

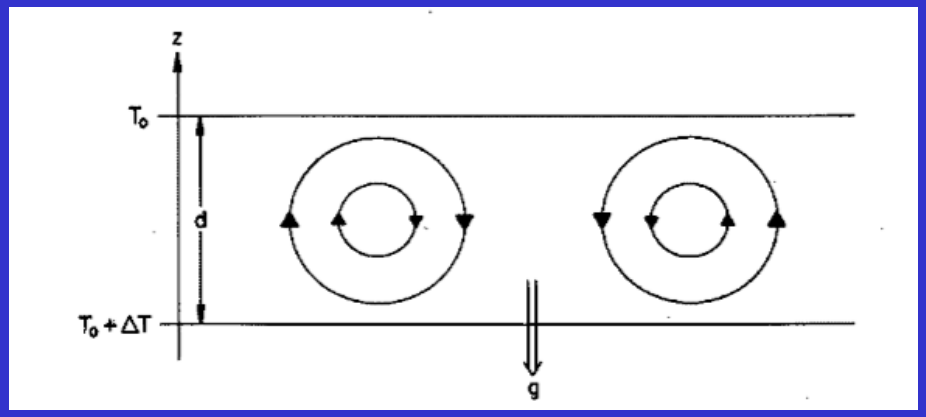
i.e. if fields strong enough, tension resists bending.

Kelvin-Helmholtz: example



Magnetoconvection

Layer of fluid heated from below:

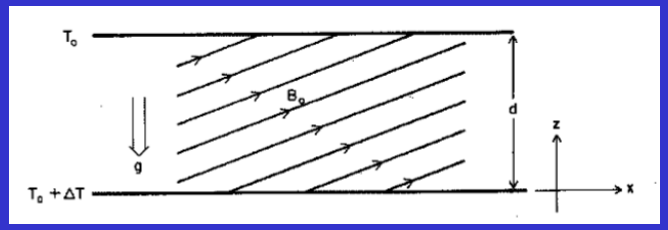


Hydro instability if temperature gradient provides enough buoyancy to lift parcel against viscous drag before thermal diffusivity can remove difference. Embodied in the Rayleigh number:

$$Ra \equiv \frac{g\alpha \Delta T d^3}{\kappa \nu} > \pi^4 \frac{27}{4} = Ra_c$$

With magnetic field (B_x and/or B_z):

Higher adverse temperature gradient needed (Ra_c is higher) since buoyant motions working against stabilising influence of tension



Vertical displacement $\xi \sim \sin(kx)$

Radius of curvature $\sim \frac{1}{k^2\xi}$ $\Rightarrow$ Magnetic tension restoring force $\sim \frac{B_0^2}{\mu} k^2 \xi$

In moving ξ, temperature decrease $= \frac{\xi \nabla T}{d}$, density decrease $= \rho_0 \alpha \frac{\xi \nabla T}{d}$, buoyancy force $= g\rho_0 \alpha \frac{\xi \nabla T}{d}$

so for buoyancy to overcome tension need $g\rho_0 \alpha \frac{\xi \nabla T}{d} > \frac{B_0^2}{\mu} k^2 \xi$ or $\boxed{g\rho_0 \alpha \frac{\nabla T}{d} > \frac{B_0^2}{\mu} k^2}$ for OVERTURNING convection

Clearly, can take place if magnetic field is weak (small $\frac{B_0^2}{\mu}$)or if the wavelength is long enough (small k^2)

Critical Rayleigh number Ra_c depends on the Hartmann number, $Ha = \frac{B_0^2}{\mu} \frac{d^2}{\rho \eta \upsilon}$

Magnetoconvection

If condition for overturning convection is not met, diffusive effects can still allow gentler convection due to the leak instability or overstability.

Can counteract the thermal and tension effects by the diffusion of heat and magnetic field. Think: resistivity allows field lines to "slip" through the fluid somewhat, reducing the effects of tension, before thermal diffusivity reduces the buoyancy force

Reduce buoyancy by factor of κ and tension by factor of η:

$$(g\rho_0\alpha\frac{\nabla T}{d})\frac{d^2}{\kappa} \text{ and } (\frac{B_0^2}{\mu}k^2)\frac{d^2}{\eta}\ .$$

Clearly, if $\eta > \kappa$ we can write this again as

$$(g\rho_0\alpha\frac{\nabla T}{d})\eta > (\frac{B_0^2}{\mu}k^2)\kappa$$

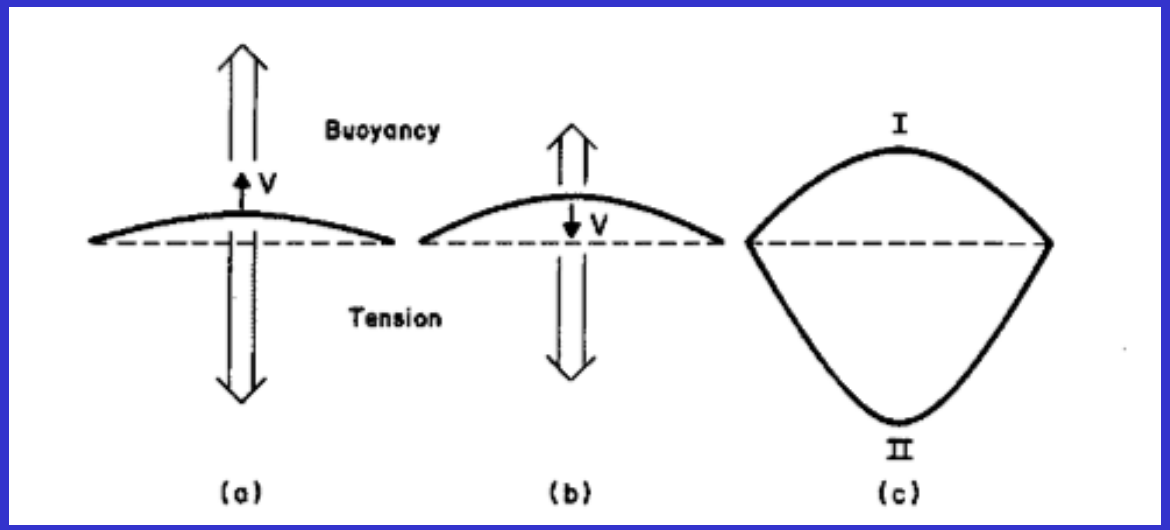
for OVERTURNING (stationary) convection

Magnetoconvection

If $P_m = \eta/\kappa < 1$: Overstable convection possible:

Overstable $\Rightarrow$ growth rate ω is complex with $\omega = \omega_r + i\omega_i$ and $\omega_r > 0$

Tension dominates over buoyancy leading to growing oscillations



- Not enough to overturn so returns to original position.
- However at this point, both tension (stabilising) and buoyancy (destabilising) have been reduced by diffusive effects and next oscillation is larger => instability

$Ra_c \sim \pi^2 Ha^2$

Flux expulsion

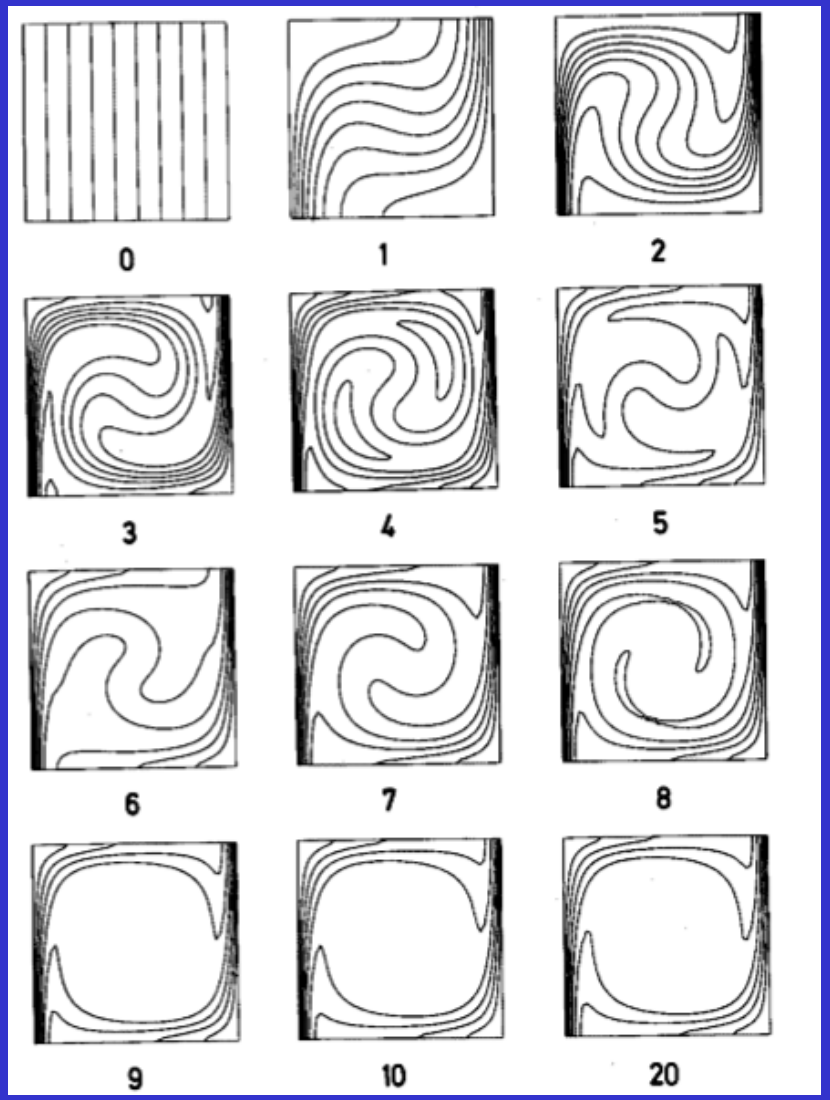
Magnetoconvection transport phenomenon:

Parker 1963a; Weiss 1966; Galloway and Weiss 1981

Weak field case so convection unhindered initially

Cells wind up magnetic field until either

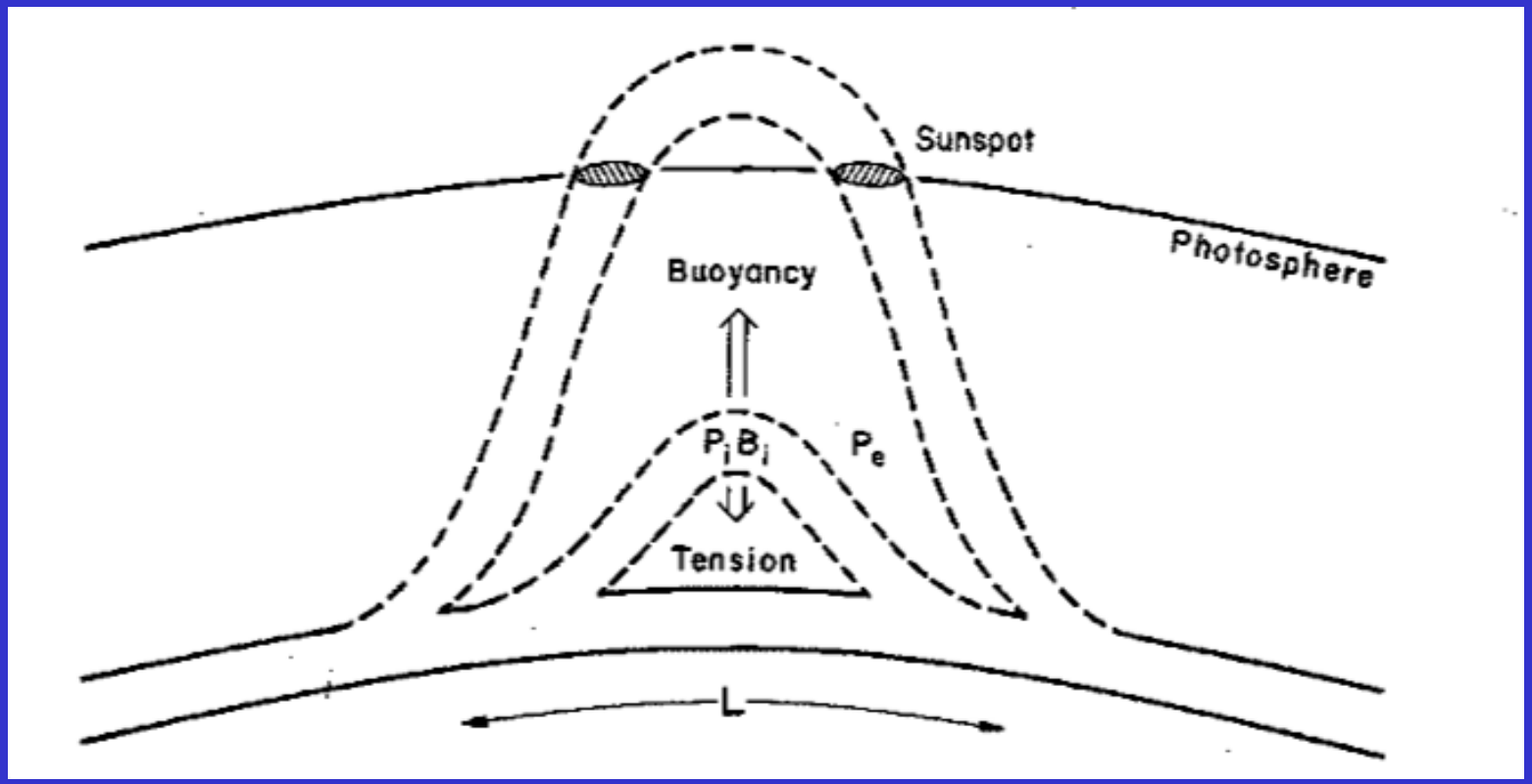
- Magnetic energy becomes comparable with kinetic energy and the flow is slowed down,
- Or, the local magnetic Reynolds number becomes order unity, and then the field lines slip through the plasma = FLUX EXPULSION



Magnetic buoyancy

Parker (1955a): Key process for sunspots and the solar dynamo. A significant transport effect for magnetic fields

Also for disks and anywhere there is stratification and magnetic field.

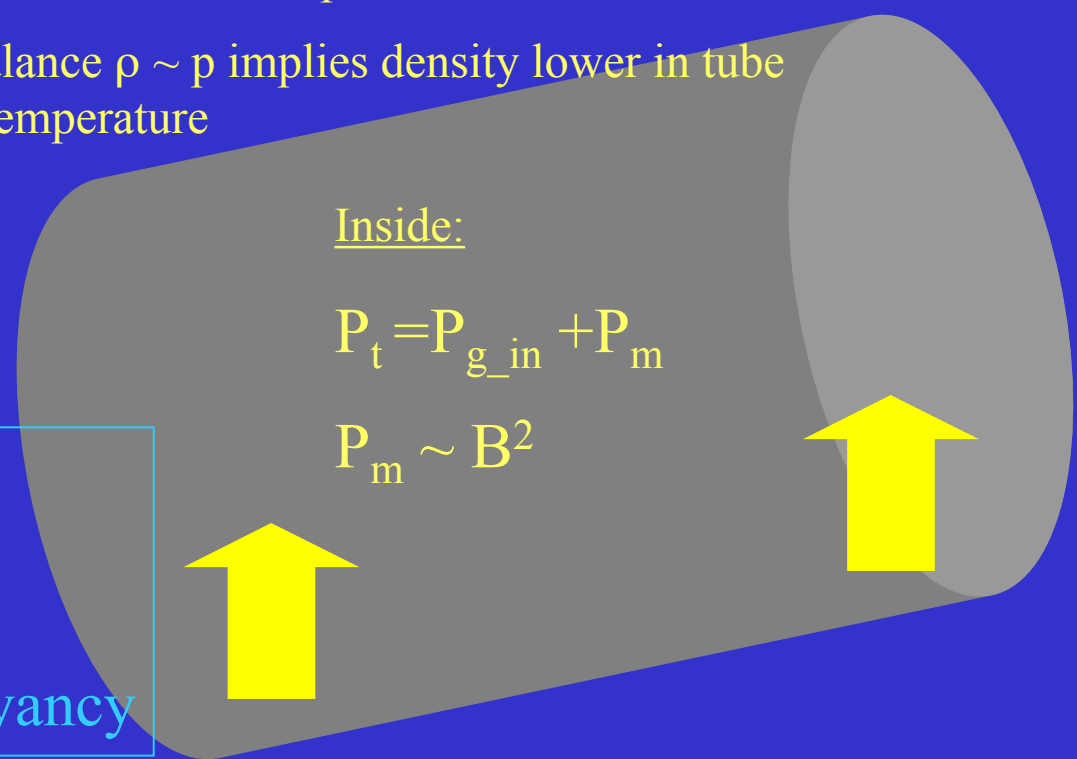


Magnetic buoyancy - non-equilibrium

MAGNETIC BUOYANCY: the standard explanation

- Magnetic field exerts a magnetic pressure (Lorenz force JxB can be split into pressure and tension $(\nabla \times \mathbf{B}) \times \mathbf{B} \sim \nabla(\mathbf{B}^2/2) + \nabla \cdot \mathbf{BB}$)
- => $P_m \sim B^2$
- Concentrated B contributes to the total pressure
- Isothermal pressure balance $\rho \sim p$ implies density lower in tube

  P_g = density*temperature



Outside:

$P_t = P_{g\_out}$

Inside:

$P_t = P_{g\_in} + P_m$

$P_m \sim B^2$

$P_t(in) = P_t(out)$

=> $P_{g\_in} < P_{g\_out}$

$\rho_{in} < \rho_{out}$ => buoyancy

Magnetic buoyancy instability

"PARCEL ARGUMENT"

Tayler (1973; Moffat 1978; Acheson 1979)

Assume adiabatic (confined dynamics) and no diffusive effects

$\Rightarrow B,\rho \propto Area \Rightarrow$

$$\frac{A+\delta A}{A} = \frac{B+\delta B}{B} = \frac{\rho+\delta\rho}{\rho} \Rightarrow \boxed{\frac{\delta B}{B} = \frac{\delta\rho}{\rho}} \quad (*)$$

$$\frac{p}{\rho^{\gamma}} = const \Rightarrow \frac{\delta p}{p} = \gamma\frac{\delta\rho}{\rho} \quad (*)$$

Total pressure $p_T = p_{gas} + p_{magnetic} = p_{gas} + \dfrac{B^2}{2\mu_0}$

Assume slow rise $\Rightarrow$ pressure equilibrium maintained

Inside parcel:

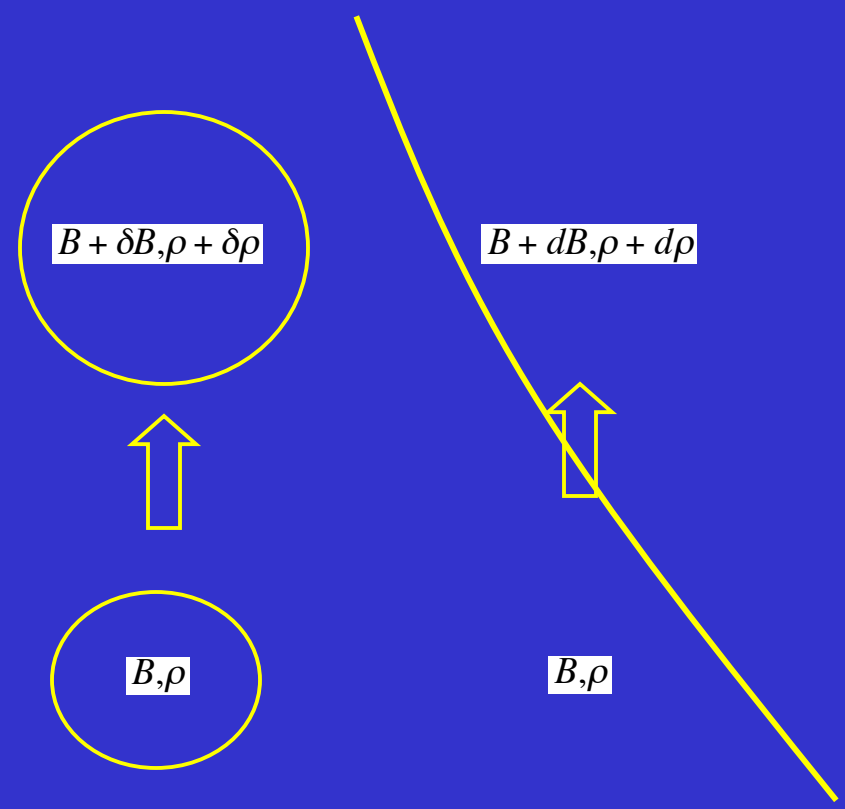
$$p_T = p + \delta p + \frac{(B+\delta B)^2}{2\mu_0} = p + \delta p + \frac{B^2 + 2B\delta B}{2\mu_0} \quad \text{(linearising)}$$

Outside parcel:

$$p_T = p + dp + \frac{(B+dB)^2}{2\mu_0} = p + dp + \frac{B^2 + 2BdB}{2\mu_0} \quad \text{(linearising)}$$

Equilibrium $\Rightarrow$ $\boxed{\delta p + \dfrac{B\delta B}{\mu_0} = dp + \dfrac{BdB}{\mu_0}}$ (*)

For instability, need $\boxed{\delta\rho < d\rho}$



Magnetic buoyancy instability

Combining the (*) relations gives

$$-g\frac{a^2}{c^2}\frac{d}{dz}\left(\ln\left(\frac{B}{\rho}\right)\right) > N^2$$

where

$$a^2 = \frac{B^2}{\mu_0\rho} \text{ (Alfven speed)}$$

$$c^2 = \frac{\gamma p}{\rho} \text{ (sound speed)}$$

$$N^2 = \frac{g}{\gamma}\frac{d}{dz}\left(\ln\left(p\rho^{-\gamma}\right)\right) \text{ (Brunt - Vaisala frequency)}$$

Alternatively :

$$\frac{B^2}{\mu_0}\left(\frac{1}{\rho}\frac{d\rho}{dz} - \frac{1}{B}\frac{dB}{dz}\right) > (stuff)$$

$\frac{B}{\rho}$ decreasing sufficiently quickly with height

$\Rightarrow$ UNSTABLE **(necessary condition)**

(i.e. destabilises a convectively stable atmosphere where $N^2 > 0$)

It's as if there is the magnetic field is supporting the gas in a "top heavy" situation $\Rightarrow$ unstable.

If don't include normal buoyancy (N), condition looks like

$$\frac{d}{dz}\left(\frac{B}{\rho}\right) > 0$$

Quick and dirty :

$\delta\rho < d\rho \Rightarrow dB < \delta B$

$\Rightarrow$

$$\frac{dB}{B} < \frac{\delta B}{B} = \frac{\delta\rho}{\rho} \Rightarrow -\frac{dB}{B} + \frac{\delta\rho}{\rho} > 0 \Rightarrow -\frac{dB}{Bdz} + \frac{\delta\rho}{\rho dz} > 0$$

Magnetic buoyancy instability

More details :

Newcomb (1961) :

$$\frac{d\rho}{dz} > -\frac{\rho g}{a^2 + c^2} \quad \text{(interchange modes)}$$

$$\frac{d\rho}{dz} > -\frac{\rho g}{c^2} \quad \text{(3D undulating modes; } k_x \to 0\text{; easier!)}$$

Thomas and Nye (1975) : equivalently

$$\boxed{-g\frac{a^2}{c^2}\frac{d}{dz}\left(\ln(B)\right) > N^2}$$

$\Rightarrow$ sufficiently fast decrease in B only is good enough now.

Including diffusion?

η small $\Rightarrow$ good, maintains B gradients

κ large $\Rightarrow$ good, erodes stabilising thermal gradients

Acheson 1979 : $k_x \to 0, \upsilon = 0$

$$-g\frac{a^2}{c^2}\frac{d}{dz}\left(\ln(B)\right) > \frac{\eta}{\kappa}N^2$$

Stars : laminar values $\frac{\eta}{\kappa} << 1$, turbulent values $\sim$ O(1)?

End result: Magnetic GRADIENTS that are important!

(gradients in the direction of gravity)

Magnetic buoyancy instability: example

Vasil and Brummell, 2008, 2009

Magnetorotational instability (MRI)

Velikhov (1959) - stability of Couette flow of hydromagnetic fluid

Chandresekhar (1960) - Global stability

Acheson and Hide (1973) - geodynamo

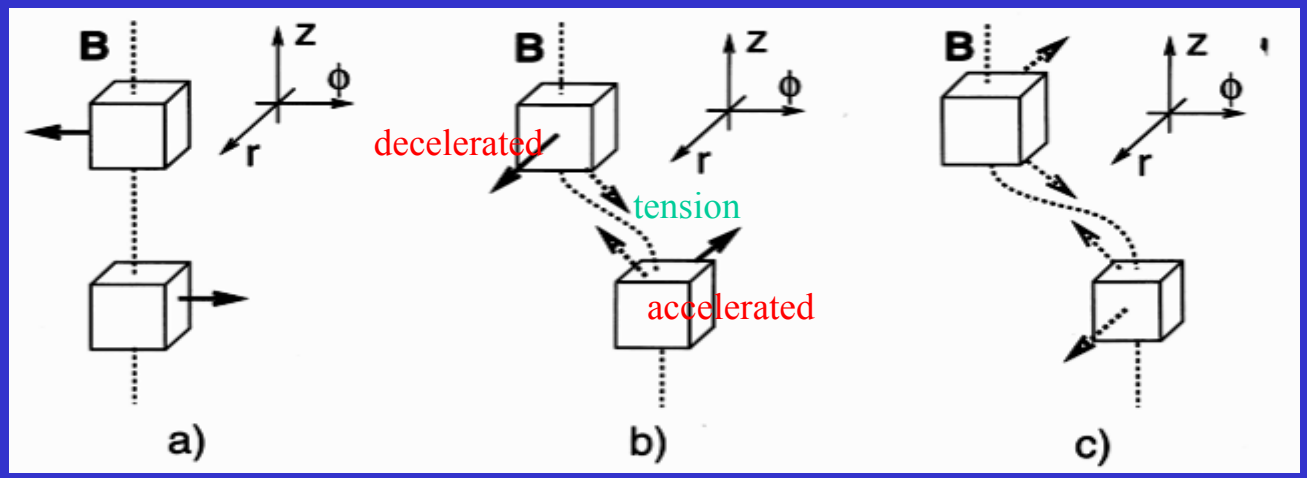
Balbus and Hawley (1991) - accretion disks

Essential instability for accretion disks: turbulent diffusivity rather than real diffusivity required to explain accretion rates. But disks are hydrodynamically stable therefore should not be turbulent. Need a magnetic instability.

Disk: Normally, stable, laminar flow if $\frac{d(r^2\Omega)}{dr} > 0$ angular momentum increases outwards (Rayleigh stability criterion)

MRI occurs if weak magnetic field is present and angular velocity decreases with radius $\frac{d\Omega}{dr} < 0$

Magnetorotational instability (MRI)



Mechanism: (axisymmetric, B_r=0 version)

- Two rings sitting above each other connected by magnetic field line (spring)
- Peturb in ϕ direction like sin(z) maintaining same velocity
- Tension acts due to stretching of magnetic field between the two
- Forward perturbation is decelerated, slows and falls inwards; backward one is accelerated, speeds up and moves out (azimuthal drag transfers angular momentum)
- Now two components of tension: azimuthal drag and radial restoring force
- For sufficiently long wavelengths (small curvature; weak tension), drag wins

Magnetorotational instability (MRI)

Good news:

- Doesn't care about toroidal field
- Incompressible is ok
- No need for self-gravity
- Ideal instability (although viscosity important as field strength -> 0)
- B_r nonzero does not change stability

All you need is weak field and angular velocity decreasing with r $\frac{d\Omega}{dr} < 0$

Previous stability criterion $\frac{d(r^2\Omega)}{dr} > 0$ replaced by MRI stability criterion $\frac{d(\Omega^2)}{dr} > 0$

Quartic dispersion relation admits other wave solutions (internal wave mode with no field and torsional long wave mode with field) but the maximum growth rate of the MRI mode is O(Ω) which is FAST (and independent of B) and the wave propogation times are SLOW.

Wavelength max growth ~ wavelength critical is proportional to B (hence very weak field => dissipation is important)

Dynamos

Perhaps the most important instability of all.

Where does the magnetic field we see everywhere come from?

Some must be relic field; some must be regenerated field

A dynamo = a magneto-fluid system that can generation and maintain a magnetic field against Ohmic diffusion

Converts mechanical energy into magnetic energy

Basic questions:

Can we generate magnetic field at least on the scale of the velocity field (or smaller)?

Can we generate magnetic field on much larger scales than the velocity field?

Kinematic theory

- Consider KINEMATIC THEORY:
- ***u*** is prescribed and cannot be altered by ***B***
- Throw away the Lorenz feedback and ignore the other equations
- Purely consider induction

$$\frac{\partial B}{\partial t} = \nabla \times (u \times B) + \eta \nabla^2 B$$

- div ***B*** = 0 (and consider div ***u*** = 0 too for convenience here)
- Notice this is now a LINEAR problem in ***B***
- ***u***=0 : clearly ***B*** must decay on timescale L/η^2
- QUESTION: what (class of) velocity fields ***u*** maintain ***B*** against Ohmic decay?

Fast dynamo theory

Pick your favourite ***u*** and solve the linear problem

Define a growth rate suitably:

$$\mathbf{u}\ \text{steady}: \ \mathbf{B}(x,t) = \hat{\mathbf{B}}(x)e^{pt} \ \text{(e - value problem)}$$

$$\mathbf{u}\ \text{periodic}: \ \mathbf{B}(x,t) = \hat{\mathbf{B}}(x,t)e^{pt} \ \text{(Floquet problem)}$$

$$\mathbf{u}\ \text{stationary random process}: \ \left\langle |\mathbf{B}|^q \right\rangle \approx e^{pt} \ \text{(moments grow exponentially)}$$

i.e. p is some measure of magnetic field growth, with p_r=Real(p)

Note that "useful" (astrophysically relevant) dynamos

- must survive at very high kinetic and magnetic Reynolds numbers
- it is no good to generate magnetic field but on an Ohmic (slow, yawn) timescale
- must be FAST (not Ohmic)

=> Definitions:

$$p_r \to \gamma > 0 \text{ as } \eta \to 0 \Rightarrow \text{ FAST DYNAMO}$$

$$p_r \to \gamma \leq 0 \text{ as } \eta \to 0 \Rightarrow \text{ SLOW DYNAMO}$$

Dynamo ingredients

Punchline: any sufficiently high Re, Rm chaotic flow will be a dynamo that generates magnetic field on the scales of the velocity field (and smaller)

Imagine a messy fluid flow that tangles things up.

Then dynamo action is a competition between two effects:

- Growth of magnetic field by stretching of field lines as they are advected by the flow (remember, for $\eta \to 0$, field lines are material lines).
- Destruction of magnetic fields by enhanced diffusion due to increased gradients for finite η

In chaotic flows, both of these effects proceed at an exponential rate, due the exponential separation of trajectories:

- Amplification happens as points separate
- Destruction happens as points come together

Can these happen at DIFFERENT rates?

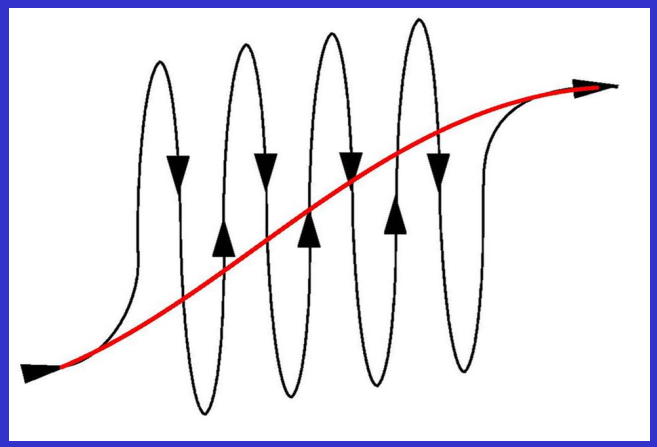
For a scalar quantity is NO. Purely advecting a scalar means that we enhance diffusion at exactly the same rate as we generate the gradients.

BUT the VECTOR equivalent has a way of reducing the enhanced diffusion and letting the stretching win:

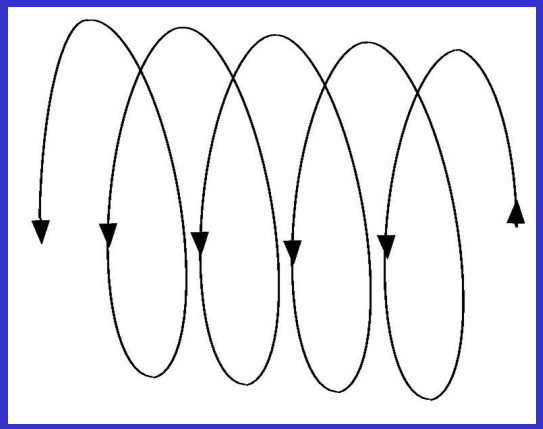
We can play with the topology of the field!

Stretching and diffusion

Topology:



Diffuses



Does not diffuse

So depends on how lines are PACKED as well for a vector field.

NOTE that this is NOT POSSIBLE IN 2D! (anti-dynamo theorems)

CONCLUSION:

GENERATION OF SMALL-SCALE FIELDS - NOT SO BAD.

… GENERATION OF LARGE-SCALE FIELDS MUCH MORE DIFFICULT PROBLEM! …

Mean field theory

- Parker 1955; Braginskii 1964a,b; Steenbeck, Krause & Radler 1966; Moffatt 1979
- Try and get around the anti-dynamo theorems (must not behave like a scalar)
- Two-scale approach: decompose field into large- and small-scale parts.
- Need a meaningful average: time, space, ensemble – easiest to think of as space
- $$\mathbf{U} = \langle \mathbf{U} \rangle + \mathbf{u}, \quad \mathbf{B} = \langle \mathbf{B} \rangle + \mathbf{b}$$
  $$\langle \mathbf{u} \rangle = \langle \mathbf{b} \rangle = 0$$
- <> means compute average over a volume larger than the typical scales of u,b but smaller than the typical scales of U,B

Mean field theory

Substitute in induction equation and take the <> average:

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times \left(\langle \mathbf{U} \rangle \times \langle \mathbf{B} \rangle \right) + \boxed{\nabla \times \left(\langle \mathbf{u} \times \mathbf{b} \rangle \right)} + \eta \nabla^2 \langle \mathbf{B} \rangle$$

$$\langle \mathbf{u} \times \mathbf{b} \rangle = \mathbf{E}$$ the average EMF

LARGE-SCALE

Subtract from the original equation:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \left(\langle \mathbf{U} \rangle \times \mathbf{b} \right) + \nabla \times \left(\mathbf{u} \times \langle \mathbf{B} \rangle \right) + \boxed{\nabla \times \left(\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle \right)} + \eta \nabla^2 \mathbf{b}$$

$$\mathbf{u} \times \mathbf{b} - \langle \mathbf{u} \times \mathbf{b} \rangle = \mathbf{G}$$ the " pain in the neck term"

SMALL SCALE

Notice now:

- Large-scale field has a source from the average emf
- Large-scale field can be 2D or axisymmetric and anti-dynamo theorems do not apply, since fluctuations may be 3D
- Even in the absence of all large-scale fields, there is a source (the G term) for the small-scales => small-scale self-excited dynamo.

Mean field theory

- BUT the large-scale equation is not CLOSED due to the E term. Need to write E in terms of <U> and <B>
- Notice that for the KINEMATIC problem, the small-scale b is a linear equation with curl(u x <B>) acting as a source term
- This implies that b is linearly related to <B> and then so is the emf.
- So write $\mathbf{E}_i = \alpha_{ij}\langle \mathbf{B} \rangle_j + \beta_{ijk}\partial_k \langle \mathbf{B} \rangle_j + \gamma_{ijkl}\partial_k \partial_l \langle \mathbf{B} \rangle_j + \ldots$
- Coefficients α,β depend on everything but <B> i.e. <U>, the statistics of u, η

Mean field theory: α,β

- α,β are PSEUDO-TENSORS relating <B> (regular/polar vector) to E (pseudo/axial vector).
- (Regular vectors switch signs on reversal of the co-ord system; pseudovectors do not)
- α,β depend on (the statistics of) u.
- Easiest assumption: assume isotropic. Then

$$\alpha_{ij} = \alpha\delta_{ij}, \quad \beta_{ijk} = \beta\varepsilon_{ijk}$$

- Then first approximation to E is: $\mathbf{E}^{(0)} = \alpha\langle\mathbf{B}\rangle$
- Thus the current is: $\mathbf{j}^{(0)} = \sigma\alpha\langle\mathbf{B}\rangle$
- This is the ALPHA EFFECT: a current is driven *parallel* to <B> (normally it is *perpendicular* due to u x b)
- So now a toroidal field drive a toroidal current which gives rise to a poloidal field.
- Regeneration of the poloidal field by the alpha effect

Interpretation: α

- Parker 1955:
- Take a straight fieldine
- Lift it up and twist it
- Small-scale fluctuations (twist) cause emf and then a current parallel to the original straight fieldline.
- If QUICK, SHORT twist, then induced current is in direction of field (otherwise might be anti-parallel)
- Ensemble of many short events regenerates field.
- Notice, all twists must be in same way, not random, to constructively combine.
- Related to PSEUDO-SCALAR nature of α. If α must switch sign on reversing co-ordinate system, so must (statistics of) u. This mean u is NOT reflectionally-symmetric i.e. has HANDEDNESS (or CHIRAL).
- Usually manifests in HELICITY $H = \langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle$ (a pseudo-scalar), hence the search for connections between α and H.

Interpretation: β

- Isotropic assumption: $\beta_{ijk} = \beta\varepsilon_{ijk}$
- Contribution to E:

$$\mathbf{E}_i^{(1)} = \beta\varepsilon_{ijk}\partial_k\langle\mathbf{B}\rangle_j = -\beta\varepsilon_{ikj}\partial_k\langle\mathbf{B}\rangle_j$$

i.e. $\mathbf{E}^{(1)} = -\beta\nabla\times\langle\mathbf{B}\rangle$

Therefore

$$\nabla\times\mathbf{E}^{(1)} = -\nabla\times\beta\nabla\times\langle\mathbf{B}\rangle = \beta\nabla^2\langle\mathbf{B}\rangle$$

- β looks like another diffusive component: TURBULENT DIFFUSIVITY

MFE: Summary

$$\frac{\partial \langle \mathbf{B} \rangle}{\partial t} = \nabla \times \left(\langle \mathbf{U} \rangle \times \langle \mathbf{B} \rangle \right) + \alpha \nabla \times \langle \mathbf{B} \rangle + (\eta + \beta) \nabla^2 \langle \mathbf{B} \rangle$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \left(\langle \mathbf{U} \rangle \times \mathbf{b} \right) + \nabla \times \left(\mathbf{u} \times \langle \mathbf{B} \rangle \right) + \nabla \times \mathbf{G} + \eta \nabla^2 \mathbf{b}$$

Large-scale magnetic field depends on α as a source term and REQUIRES lack of reflectional symmetry in the velocity fluctuations.

Small-scale field can clearly be self-supporting with the G term as a source.

<u>To proceed with MFE:</u>

- Guess α,β and see what happens (VERY POPULAR!)
- Compute α,β rigorously in terms of u. Requires simulations or possible analytically in a very small number of circumstances:
  - o FIRST ORDER SMOOTHING: throw out the G term. Implies NO small-scale dynamo, and low Rm or short correlation time (random waves)
  - o LAGRANGIAN: throw out the diffusion term. Implies high Rm (good)

MFE: Summary

- 2D axisymmetric case in poloidal toroidal form:

$$\frac{\partial A}{\partial t} + (\mathbf{u}_P \cdot \nabla A) = \alpha B + (\eta + \beta)\nabla^2 A$$

α effect saves the day and provides a source of toroidal to poloidal

$$\frac{\partial B}{\partial t} + \mathbf{u}_p \cdot \nabla B = \mathbf{B}_p \cdot \nabla u_z + \alpha \nabla \times \mathbf{B}_p + (\eta + \beta)\nabla^2 B$$

Ω Effect: shear generation of toroidal field from poloidal

α^2 effect

- Linear equations: either grow or decay
- No nonlinear saturation: Could include through e.g. α=α (A,B)

Serious questions

- Are the concepts of α,β right? Does the micro-physics support these ideas?
- What happens in the NON-KINEMATIC (DYNAMIC) case? What is the role of the Lorenz force? How do α,β change in these circumstances?
- What happens when the small-scales ARE self-excited (G term is important i.e there is a small-scale dynamo)? (High Reynolds numbers.) Another mechanism for generating fluctuating b other than tangling of the large-scale field …
- Mean field models that explicitly construct α either have $\alpha \rightarrow 0$ as $\eta \rightarrow 0$ or have questionable validity as $\eta \rightarrow 0$. DO MEAN FIELD MODELS OPERATE IN THE ASTROPHYSICAL REGIME?

The end