(b)

Quasi-linear Theory for 1 D Plasma

- simplest mean-field theory
- fluctuation-driven diffusion of mean $\langle f \rangle$
- describe response of distribution to fluctuations

i.e. Fluctuation-driven flux .....

- application → turbulent resistivity..

e.x. Sagdeev + Galeev, "Nonlinear Plasma Theory"

also Moffatt; "Mean Field Electrodynamics"

Quasilinear Theory - Vlasov Plasma

i) Motivation and Overview

→ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv} = 0$

⇒ growth/damping rate $\gamma_k = \gamma_k[\langle f\rangle]$

but $\langle f\rangle$ evolves... If $\langle f\rangle$ evolves slowly;

ie "slowly" ⇒ $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t)\rangle]$ → {evolution driven by instabilities

⇒ Physics: mean distribution evolution .... driven by relaxation.

→ [quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f\rangle$...]

③ quasilinear theory is "mindless mean field theory", i.e.

$\langle f \rangle = \langle f(v,t) \rangle$ where → $\langle\ \rangle$ eliminates spatial dependence → f understood "slow"

so if:

$$\left[\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m} E \frac{\partial f}{\partial v} = 0 \right]$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} \left\langle \frac{q}{m} \tilde{E} \tilde{f} \right\rangle = 0$$

i.e. generic mean field equation (for $\langle f \rangle$) for mean of conserved order parameter ($\langle f \rangle$)

$$\frac{\partial \langle f \rangle}{\partial t} + \frac{\partial}{\partial v} \overline{J}_v = 0$$ → phase space continuity equation

$J_v = \Gamma_v = \left\langle \frac{q}{m} E f \right\rangle$

current

$= \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$

for: $E = \tilde{E}$

$f = \langle f \rangle + \tilde{f}$

elementary closure problem! i.e. relate $\langle f \rangle$ to $\langle \tilde{E} \tilde{f} \rangle$ → hierarchy!

How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(f) $\tilde{f} \to \tilde{f}_{Linear}$ (i.e. linear response of f)
plug in linear response

i.e. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial \langle f\rangle}{\partial v}$$

so $$J_v = -\frac{q^2}{m^2}\sum_{k,\omega}|\tilde{E}_{k,\omega}|^2\frac{i}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only) i.e. contrast approach to criticality in usual phase transition (2nd order)
Q.L. equation is:
→ here growth of order parameter in broken symmetric phase ... not noise driven

$$\left.\begin{aligned} \frac{\partial\langle f\rangle}{\partial t} &= \frac{\partial}{\partial v}D\frac{\partial\langle f\rangle}{\partial v} \\ D &= \frac{q^2}{m^2}\sum_k|\tilde{E}_k|^2\frac{i}{\omega - kv}\end{aligned}\right\}$$ "Q.L. equation"

i.e. mindless mean field theory ...

with $$\left.\begin{aligned}\epsilon(k,\omega) &= 0 \\ \partial_t|E_k|^2 &= 2\gamma_k|E_k|^2\end{aligned}\right\} \to \text{advance fields.}$$

But,
Surprisingly : Q.L.T. works quite well!

Key issue : why?

{ N.B.: In contrast to critical phenomena, external noise ignored → instability driven ... deterministic

④ Some questions to keep in mind :

i) why is QL equation a diffusion equation? When is this valid?

↔ nature of "irreversibility" ....

ii) can Q.L. equation be derived from Fokker-Planck theory?

↔ also "irreversibility" related ...

iii) how does Q.L. equation balance the energy-momentum budgets?

iv) When / how } does Q.L. theory fail?

↔ related (i) ...., What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated?

v) what is dynamics of quasilinear relaxation?

i.e. physics?

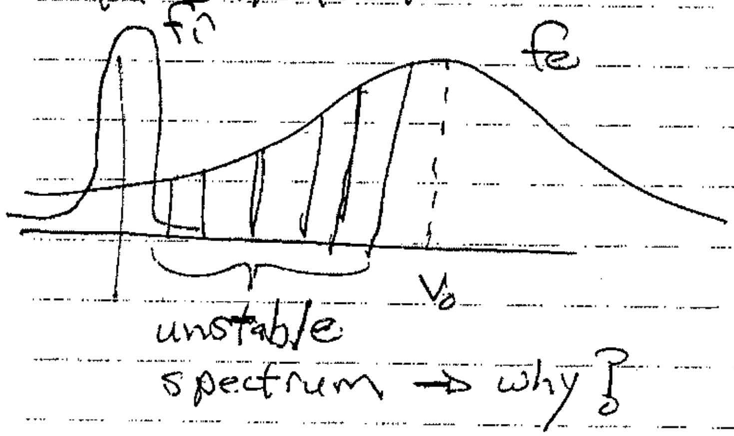
iii) Basic Scales / Regime Definition

① → Generally, Q.L.T. concerned with

i.) 'broad' spectrum of:

ii.) unstable waves

i.e. for current-driven ion-acoustic (C.D.I-A.) turbulence:



②
⇒ In finite system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$$\omega_m / k_m = \omega(k_m)/k_m = V_{ph\,m}$$

- wave-particle resonance occurs when

$$V = V_{ph,m}$$

then Sir Isaac ⇒

$$m\ddot{x} = \sum_m q E_m \cos(k_m x - \omega_m t)$$

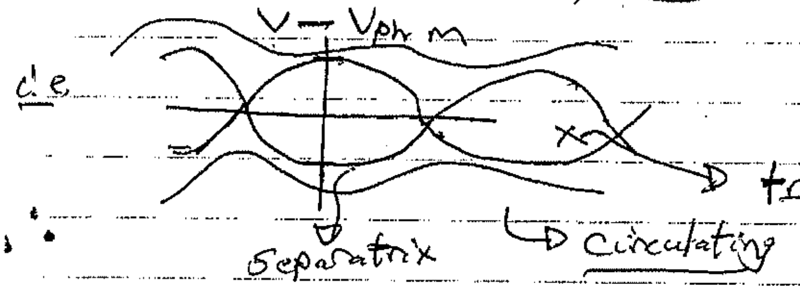
n.b. { deterministic, no RPA

and 1 resonance dominant ⇒

$$m\ddot{x} \cong q E_{m_0} \cos(k_m x_0 + (k_m v - \omega_m) t)$$

⇒ each resonant velocity defines a phase space island

i.e.

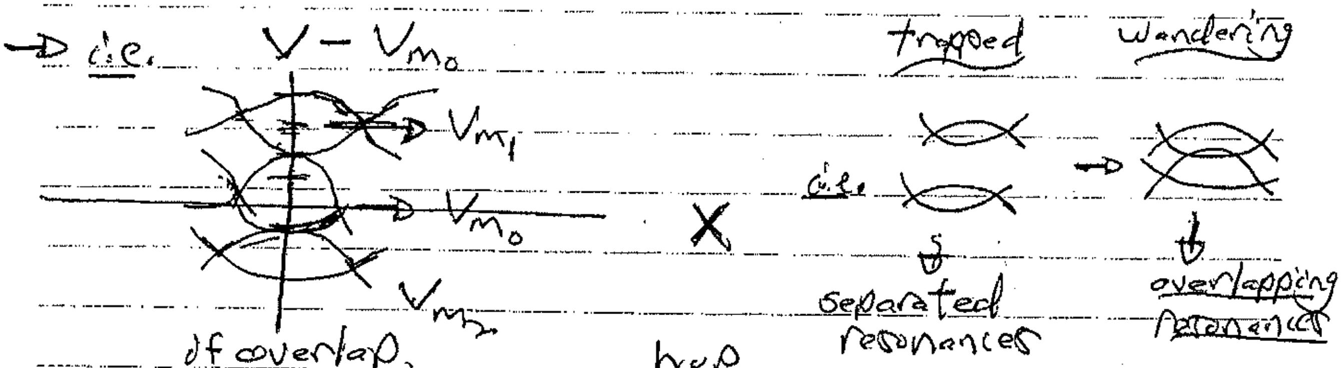


$$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$$

QLT is concerned with the case of:

⇒ multiple, overlapping resonances → { separatrix proximity ⇒ destruction

⇒ i.e.



∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v, $D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$ ↳ what is it?

Δv → resonance width

τ_{ac} → pattern time

Overlap condition (B.V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \gtrsim V_{ph_{m+1}} - V_{ph_m}.}$$

$$\Delta V_T \sim \sqrt{e\phi/m}$$

∴ → particle motion stochastic

→ fundamental irreversibility ⇒ orbit stochasticity (not dissipation, Landau damping ... ⟺ contrast critical phenomena).

→ underpinning of diffusion equation.

③ → But, a swindle !? → use of un-perturbed orbit in estimate!

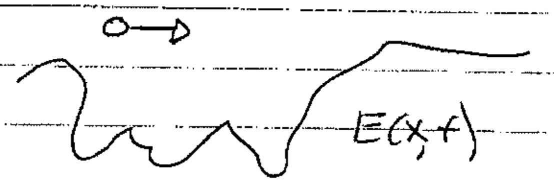
i.e. is $x \to x_0 + vt$ valid ?

Consider: linear, un-perturbed orbit ?!

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

∴ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern

obviously, ① $\tau_L \ll \tau_b \rightarrow$ unperturbed orbit is satisfactory approximation $\sim QL$

(pattern changes prior $\leftrightarrow$ bouncing)

② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes, $\sim NL$

so must consider orbit perturbation...

i.e.

$\tau = 0$ $\qquad\qquad$ τ_b

v.s.

∴ quasilinear theory relevant to evolution when:

① $\rightarrow$ orbits stochastic (Chirikov condition satisfied)

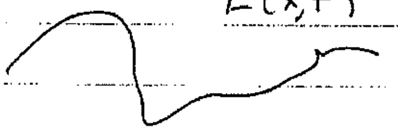
② $\rightarrow$ $\tau_{Life} < \tau_{bounce}$ $\rightarrow$ unperturbed orbits ≈ valid.

3)

But, how relate $\tau_{lifetime}$, τ_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$E(x,t)$



$$\Leftrightarrow \sum_{k} E_k e^{i(kx-\omega_k t)}$$

$$= \sum_{k} E_k \exp\left[i\left(k\left[x-\underbrace{\omega_k/k}_{v_{ph}(k)}\, t\right]\right)\right]$$

$\therefore$ $\underbrace{\Delta(\omega_k/k)}_{\text{sets dispersal rate}} \equiv$ spread in phase velocities.

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength

$$1/\tau_{life} \equiv k\,\Delta(\omega_k/k)$$

$$= k\left(\underbrace{\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k}_{(v_g \;-\; v_{ph})}\right)$$

$$= \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k \;=\; \big(v_g(k) - v_{ph}(k)\big)\,\Delta k$$

n.b. $\begin{cases} \tau_{life} \to \infty \text{ for non-dispersive waves!} \\ \text{Generally; QLT / weak turbulence encounters trouble for } \begin{cases}\text{non dispersive} \\ \text{weakly dispersive}\end{cases} \text{ waves.} \end{cases}$