generally, shorter time dominates, except for non-dispersive waves.

So, can enumerate key time scales

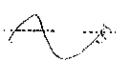
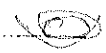
$\tau_{ac} = |\Delta k (v_{ph} - v_{gr})|^{-1}$

$\equiv$ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles.

γ^{-1} = growth/damping time

$\tilde{\tau}_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1} \equiv$ trapping time

$\tilde{\tau}_{relax} = \left(\frac{1}{\langle f \rangle}\frac{\partial \langle f \rangle}{\partial t}\right)^{-1} \equiv$ avg. distribution relaxation time

So  

$\tilde{\tau}_{ac} < \tilde{\tau}_{Tr} \Rightarrow$ U.P.O. valid

$\tilde{\tau}_{ac}, \gamma^{-1} < \tilde{\tau}_{relax} \Rightarrow \langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tilde{\tau}_{relax} \Rightarrow$ Q.L.T. valid.

iii.) Energy – Momentum Budgets

→ Key Point: There are two ways of implementing the book-keeping and accounting

i.e. $\begin{cases} \text{resonant particles} & \text{vs. 'waves'} \\ \underline{\text{or}} & \\ \text{particles} & \underline{\text{vs.}} \text{ fields} \end{cases}$

keep in mind: Wave = Field + Non-resonant Particles

i.e. for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial\omega}(\omega\epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega\frac{\partial\epsilon}{\partial\omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle.

(Show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f \rangle}{\partial t} = -\frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$\frac{\partial}{\partial t} \int dv \, \frac{mv^2}{2} \langle f \rangle = -\int dv \, \frac{mv^2}{2} \frac{\partial}{\partial v} \frac{q}{m} \langle \tilde{E} \tilde{f} \rangle$$

$$= \int dv \, \cancel{m} v \frac{q}{\cancel{m}} \langle \tilde{E} \tilde{f} \rangle$$

Plugging in $\tilde{f}_k^{\text{Linear}}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t} \mathcal{E}_{kin} = -i \int dv \, \frac{v q^2}{m} \sum_k |E_k|^2 \left(\frac{P}{\omega - kv} - i\pi \delta(\omega - kv) \right) \frac{\partial \langle f \rangle}{\partial v}$$

$$\frac{\partial}{\partial t} \mathcal{E}_{kin}^{res} = -\int dv \, \frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \delta(\omega/k - v) \frac{\partial \langle f \rangle}{\partial v} |E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m} \sum_k \frac{\omega}{k|k|} \left. \frac{\partial \langle f \rangle}{\partial v} \right|_{\omega/k} |E_k|^2$$

As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

· In wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$

$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega}$ $\qquad \cancel{i}\gamma_k = -\cancel{i}\frac{\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$

$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$

Now, $W \equiv$ Wave Energy Density

$$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi}$$

$$= \sum_k \omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi}$$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k\, \omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k} \frac{|E_k|^2}{8\pi} \qquad |E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$= \sum_k 2\left(\frac{-\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k \frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\epsilon_{IM} = \frac{\omega_p^2}{k}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore \frac{\partial W}{\partial t} = \sum_k \pi \frac{q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\bigg|_{\omega/k}|E_k|^2$$

$$\Rightarrow \boxed{\partial_t \mathcal{E}_{kinetic}^{resonant} + \partial_t W = 0}$$

$\cancel{\partial_t \int dv\, \frac{1}{2}mv^2 f = q\langle E\tilde{f}\rangle}$

Note:

- this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\underbrace{\frac{\partial}{\partial t}W}_{\text{wave energy}} + \underbrace{\underline{\nabla}\cdot\underline{S}}_{\text{divergence of wave energy density flux}} + \underbrace{Q}_{\langle\underline{\tilde{E}}\cdot\underline{\tilde{J}}\rangle \text{ coupling}} = 0$$

For homogeneous system: $\underline{\nabla}\cdot\underline{S} = 0$

so $\dfrac{\partial W}{\partial t} + Q = 0$

Q: $\langle E\cdot J\rangle$ mediated by resonant particles (DC field)

$$\boxed{\Leftrightarrow \quad \frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(RPKED) = 0}$$

RPKED: resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$W = NRPKED + FED$

NRPKED: non-resonant particle kinetic energy density ✓

FED: field energy density ✓

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}(\underbrace{RPKED + NRPKED}_{PKED}) = 0$$

So $\boxed{\frac{\partial}{\partial t}\,\text{FED} + \frac{\partial}{\partial t}\left(\text{PKED}\right) = 0}$ Energy Thm.

i.e. fields and particles conserve energy.

What is the Physics of all this ??

$$D = \mathrm{Re}\sum_k \frac{q^2}{m^2}|E_k|^2 \left(i/\omega - kv\right)$$

QL diffusion

for general, weakly non-stationary state ---

$$= \sum_k \frac{q^2}{m^2}|E_k|^2 \left(|\gamma_k| \big/ \left((\omega-kv)^2 + |\gamma_k|^2\right)\right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\cong \sum_k \frac{q^2}{m^2}|E_k|^2 \Big\{ \underbrace{\pi\delta(\omega - kv)}_{\text{resonant diffusion}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\text{non-resonant diffusion}} \Big\}$$

Resonant Diffusion → irreversible – resonance overlap is underpinning

→ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy ... ?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2} |E_k|^2 \frac{\gamma_k}{\omega_k^2}$$

ponderomotive energy

$$= \frac{1}{2} \partial_t \sum_k |V_k|^2 \qquad \text{where } |V_k|^2 = \frac{q^2}{m^2} \frac{|E_k|^2}{\omega_k^2}$$

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \, \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka "fake diffusion"

→ vanishes in stationary state

Point is that can count non-resonant diffusion as:

$$D_{\text{non-resonant}} \begin{cases} \dfrac{\text{particle sloshing energy}}{\text{part of } \underline{\text{wave}} \text{ energy density}} \\ \text{or} \\ \text{part of total particle kinetic energy density} \end{cases}$$

so $\underline{\underline{\text{two}}}$ forms of energy conservation!

Note: Physically, the picture of plasma as

gas $\begin{cases} - \text{resonant particles} \\ - \text{waves} \end{cases}$ $\underline{\underline{\text{or}}}$ equivalently

resonant particles + quasi-particles

$$\text{quasi-particles} \rightarrow \text{waves} \begin{cases} N(k, x, t) \\ WKE, \text{ etc.} \end{cases}$$

is appealing and will pervade this course.

N.B.: Direct Proof of $\partial_t (PKED + FED) = 0$

From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv \, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int \frac{dv}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \, \frac{\omega_p^2}{k} (\underbrace{kv - \omega}_{} + \omega) * \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$\hookrightarrow$ {cancels denom, residual odd in k}

$$= -i \sum_k \frac{|E_k|^2}{4\pi} \int dv \, \frac{\omega_p^2}{k} \frac{\omega}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi} \omega_k$$

$$\omega_k = \omega_k^r + i\gamma_k$$

$$= -\sum_k \frac{|E_k|^2}{8\pi} (2\gamma_k)$$

$$= -\partial_t (FED) \quad \checkmark$$