From kinetic theory to microscopic transport

Douglas Gough

Start easy: shout if you're bored, and I could speed up.

In true equilibrium everything is uniform & steady. Fluid satisfies statistical mechanics with which you are all familiar. I shall be discussing what happens when there are small departures from uniformity on length and timescales much greater than interparticle distances and collision times, so that I can address small departures from statistical (thermodynamic) equilibrium.

Distribution function $f(\underset{\sim}{x}, \underset{\sim}{p}, t)$

Simple theory: $\underset{\sim}{p}$ could be extended to include magnetic moment, state of excitation.

Assume all particles identical & mass m. $\underset{\sim}{p} = m\underset{\sim}{v}$ nonrelativistic

$$f\, d^3x\, d^3p \;=\; \text{no. of particles in } d^3x\, d^3p \text{ about } \underset{\sim}{x}, \underset{\sim}{p}. \qquad \text{6-d phase space}$$

Binary collisions randomize distribution. So do higher-order collisions which, for short-range interactions dominate in a dilute gas.

Rate of collision of 2 particles from $\underset{\sim}{p}, \underset{\sim}{p}_1$ to $\underset{\sim}{p}', \underset{\sim}{p}_1'$ is $S(\underset{\sim}{p}', \underset{\sim}{p}_1'; \underset{\sim}{p}, \underset{\sim}{p}_1)\, d^3p\, d^3p_1\, d^3p'\, d^3p_1'$

(S for ~~...~~ scattering)

This can be related to scattering cross section $d\sigma$:

$$|\underset{\sim}{v} - \underset{\sim}{v}_1|\, d\sigma = S\, d^3p'\, d^3p_1' \qquad (1)$$

$d\sigma$ is determined by scattering theory

Smooth fn subject to momentum & energy being conserved

Scattering is elastic (reversible in t) — no excited states

Time reversal: $\underset{\sim}{p}' \to -\underset{\sim}{p} = \underset{\sim}{p}^t$

$$S(p', p_1'; p, p_1) = S(\underset{\sim}{p}^t, \underset{\sim}{p}_1^t; \underset{\sim}{p}'^t, \underset{\sim}{p}_1'^t)$$

For monatomic particles orientation irrelevant. $\therefore$ reverse collision has same rate

$$S(\underset{\sim}{p}', \underset{\sim}{p}_1'; \underset{\sim}{p}, \underset{\sim}{p}_1) = S(\underset{\sim}{p}, \underset{\sim}{p}_1; \underset{\sim}{p}', \underset{\sim}{p}_1')$$

Write as $\underline{S = S'}$

Equilibrium

Then statistically each collision (type) is balanced by reverse collision (type) in equilibrium

$$S f_0 f_{01}\, d^3p\, d^3p_1\, d^3p'\, d^3p_1' = S' f_0' f_{01}'\, d^3p\, d^3p_1\, d^3p'\, d^3p_1' \qquad (2)$$

where f_0 = equilib. distribn fn = const. $e^{-\varepsilon(p)/kT}$ $\qquad \varepsilon = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Maxwell – Boltzmann distribution

[define const. later when I specify indep. variable ($\underset{\sim}{v}$, not $\underset{\sim}{p}$)

It is indep of $\underset{\sim}{p}$ and $\underset{\sim}{x}$

Elastic $\qquad \varepsilon + \varepsilon_1 = \varepsilon' + \varepsilon_1'$

$$\therefore \quad f_0 f_{01} = f_0' f_{01}' \qquad (3)$$

Eq (1) is a property of quantum scattering — matrix S is unitary. It's no more than what I've just said, but some people (mistakenly) feel happier if it's quantum: but it comes from no more than what I've just said.

Non-equilibrium

If collisions were absent the distrib^n f^n would not change following the particles in d^3p:

$$\frac{df}{dt} = 0$$

~~In presence of collisions~~ ie. $\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx_i}{dt}\frac{\partial f}{\partial x_i} + \frac{dp_i}{dt}\frac{\partial f}{\partial p_i} = 0$ Liouville's Theorem

In presence of collisions: $$\frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} + F_i\frac{\partial f}{\partial p_i} = \underbrace{C(f)}_{\text{collision integral}} \qquad (4)$$

because it comes from integrating over all collisions

Note that $v_i = \frac{p_i}{m}$ is indep of x_i in 6-d phase space so can take v_i inside derivative

Same with F_i if indep of p_i (e.g. conservative force) — not so of Lorentz force. (Pat Diamond will address)

$$\therefore \quad \frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(v_i f) + \frac{\partial}{\partial p_i}(F_i f) = C(f) \qquad (4)$$

Boltmann transport equation

The collision integral

Rate of scattering out of d^3p per unit volume is $d^3p\int S f f_1\, d^3p_1\, d^3p'\, d^3p_1'$

" " " into d^3p " " " " $d^3p\int S' f' f_1'\, d^3p_1\, d^3p'\, d^3p_1'$

Nb in equilibrium these are the same

However $S' = S$ because this is just about the scattering process

$$\therefore \quad \boxed{\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_i}(v_i f) + \frac{\partial}{\partial p_i}(F_i f) = \int S'(f'f_1' - f f_1)\, d^3p_1\, d^3p'\, d^3p_1'} \qquad (5)$$

Boltzmann's Equation

I have assumed that no. particles actually undergoing collisions is negligible. Collisions are short-lived and spatially localized cf timescale & lengthscales over which variation of macroscopic fluid is considered. We'll see that this is actually a delicate assumption.

Boltzmann's H theorem

This is background – not central to details of what I am about to develop, except to support an approximation.

Nb $\quad \frac{\partial}{\partial\alpha}(f\ln f) = (\ln f + 1)\frac{\partial f}{\partial\alpha}$ for some variable α

$$H = \int f \ln f \, d^3p$$

H is a measure of randomness $\propto -s$ (specific entropy)

$$\frac{dH}{dt} = \int \frac{\partial f}{\partial t}(1+\ln f)\,d^3p$$

If $\frac{\partial f}{\partial t} = 0$, then $\frac{dH}{dt} = 0$

$$= -\int \left[v_i\frac{\partial f}{\partial x_i} + F_i\frac{\partial f}{\partial p_i} - C(f)\right](1+\ln f)\,d^3p$$

$$= -\int \left(v_i\frac{\partial}{\partial x_i} + F_i\frac{\partial}{\partial p_i}\right) f\ln f\,d^3p + \int C(f).(1+\ln f)\,d^3p$$

$$= \underbrace{-\int\left[\frac{\partial}{\partial x_i}(v_i f\ln f) + \frac{\partial}{\partial p_i}(F_i f\ln f)\right]d^3p}_{=0} + \int C(f)(1+\ln f)\,d^3p$$

$$= \int S'(1+\ln f)(f'f_1' - ff_1)\,d^3p\,d^3p_1\,d^3p'\,d^3p_1'$$

$$= \tfrac{1}{2}\int S'[2+\ln(ff_1)](f'f_1' - ff_1)\,d^3p\,d^3p_1\,d^3p'\,d^3p_1'$$

Nb $\frac{dH}{dt}$ in equilibrium

interchanging the two (identical) incident particles

Interchanging $\{\underset{\sim}{p}, \underset{\sim}{p}_1\}$ with $\{\underset{\sim}{p}', \underset{\sim}{p}_1'\}$ makes no change

$$= \tfrac{1}{2}\int S[1+\ln(f'f_1')](ff_1 - f'f_1')\,d^3p'\,d^3p_1'\,d^3p\,d^3p_1^*$$

$$= -\tfrac{1}{2}\int S\,\ln\left(\frac{ff_1}{f'f_1'}\right)\cdot\left(\frac{ff_1}{f'f_1'} - 1\right) f'f_1'\,d^3p\,d^3p_1\,d^3p'\,d^3p_1'$$

$(x-1)\ln x > 0 \qquad (x>0)$

$$\leqslant 0$$

$$\frac{dH}{dt} = 0 \quad \text{in equilibrium}$$

If $\frac{dH}{dt} = 0$, then $f_1 f = f_1' f'$

This is just equation (3).

Out of equilibrium H will naturally <u>decrease</u> to its equilibrium value

Could linearize f about f_0 : rate of decrease $\propto \delta f = f - f_0 \implies$ exponential relaxation.

If $\exists$ gradient then there must be a departure from equilibrium. Idea is to estimate that by linearising f about f_0. But before that I'll introduce some simple concepts, by way of relaxation to let what I've just told you sink in.

Integrate equ^n (1) over $d^3p' \, \& \, d^3p_1'$ to give total scattering cross section σ

If n is number density of particles $n\sigma \ell = 1$ defines ℓ = length of cylinder to fully "block" it

$$\ell = \text{mean free path}$$

$$n = \int f \, d^3p$$

$$\tau = \ell / \bar{v} = \text{mean collision time}$$

<u>Macroscopic equations</u>

f varies over space <u>gradually</u>, on lengthscale $L \gg \ell$

$$n = \int f \, d^3p$$

$$\underset{\sim}{u} = n^{-1} \int \underset{\sim}{v} f \, d^3p \equiv \langle \underset{\sim}{v} \rangle \qquad \text{which defines } \langle \; \rangle$$

$$e = m^{-1} n^{-1} \int \varepsilon f \, d^3p = \frac{\langle \varepsilon \rangle}{m} \qquad \varepsilon = \tfrac{1}{2} m v^2$$

defined this way because thermodynamics deals with quantities <u>per unit mass</u>

<u>Collisions</u> do not change these quantities.

NB $\rho = mn$ density of fluid

$$\therefore \quad \int C(f) \, d^3p = 0, \quad \int \underset{\sim}{v} \, C(f) \, d^3p = 0, \quad \int \varepsilon \, C(f) \, d^3p = 0 \qquad \text{—— (6)}$$

Now multiply Boltzmann transport equation by m, p_i and ε and integrate over d^3p.

<u>Mass</u>

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} \underbrace{\int m v_i f \, d^3p}_{\rho u_i} + m \underbrace{\int \frac{\partial}{\partial p_i} (F_i f) \, d^3p}_{=0} = \int C(f) \, d^3p = 0$$

i.e.

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \qquad \text{—— (7)}} \qquad \text{mass conservation}$$

Momentum

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \Pi_{ij}}{\partial x_j} + \int p_i \frac{\partial}{\partial x_j}(F_j f) d^3p = 0$$

same analysis as $\frac{\partial}{\partial x_i}(\rho u_i)$ in previous equation

$\parallel$ integrate by parts

$$-\int \delta_{ij} F_j f d^3p = -F_i \int f d^3p$$

$$\therefore \quad \boxed{\frac{\partial \rho u_i}{\partial t} = -\frac{\partial \Pi_{ij}}{\partial x_j} + \frac{\rho}{m} F_i \quad \text{— (8)}}$$

where $\underline{\Pi_{ij} = m\int v_i v_j f d^3p}$ — (9)

Energy

$$\frac{\partial}{\partial t}\int \varepsilon f d^3p + \frac{\partial}{\partial x_i}\int \varepsilon v_i f d^3p + \int \varepsilon F_j \frac{\partial f}{\partial p_j} d^3p = 0$$

$$\frac{\partial}{\partial t}(\rho\epsilon) \qquad \frac{\partial q_i}{\partial x_i} \qquad \int\left[\frac{\partial}{\partial p_i}(\varepsilon F_i f) - f F_i \frac{\partial \varepsilon}{\partial p_i}\right] d^3p$$

$$\varepsilon = \frac{p^2}{2m} \qquad \frac{\partial \varepsilon}{\partial p_i} = \frac{p_i}{m} = v_i$$

$$\underline{q_i = \int \varepsilon v_i f d^3p} \quad \text{— (10)}$$

$$\frac{\partial}{\partial p_i}(\varepsilon F_i f) \text{ term } = 0, \qquad -f F_i \frac{\partial \varepsilon}{\partial p_i} \text{ term } = -F_i \int f v_i d^3p = -n \underset{\sim}{u} \cdot \underset{\sim}{F}$$

$$\boxed{\frac{\partial \rho\epsilon}{\partial t} + \frac{\partial q_i}{\partial x_i} - \rho \underset{\sim}{u} \cdot \frac{\underset{\sim}{F}}{m} = 0 \quad \text{—— (9)}}$$

It is normal to replace $\underset{\sim}{F}$ by $m^{-1}\underset{\sim}{F}$

Force per unit mass.

Now refer velocities to frame moving with the (macroscopic) fluid

$$v_i = u_i + v_i' \qquad \langle v_i' \rangle = 0$$

$$\Pi_{ij} = m\int (u_i + v_i')(u_j + v_j') f d^3p = \rho u_i u_j + m\int v_i' v_j' f d^3p$$
$$= \rho\,(u_i u_j + \langle v_i' v_j' \rangle)$$

In equilibrium $\langle v_i' v_j' \rangle$ isotropic $= \frac{1}{3}\langle v'^2 \rangle \delta_{ij}$ where $\underline{v' = |\underset{\sim}{v}'|}$

$$\langle v'^2 \rangle = \frac{3kT}{m} \qquad nkT = p = \text{pressure}$$

$$= 3\frac{p}{\rho}$$

NOT momentum of a particle

$$\therefore \quad \underline{\Pi_{ij} = \rho u_i u_j + p\delta_{ij}} \qquad \text{———} \quad (11)$$

But when flow is sheared the particle distrib^{n} is not in equilibrium such as to be Maxwellian (see viscosity below)

$$\int \epsilon v_i f \, d^3p = \int \tfrac{1}{2} m (u^2 + 2\underset{\sim}{u}\cdot\underset{\sim}{v}' + v'^2)(u_i + v_i') f \, d^3p$$

$$= \int \tfrac{1}{2} m (u^2 + v'^2) u_i f \, d^3p + \int m \underset{\sim}{u}\cdot\underset{\sim}{v}' v_i' f \, d^3p$$

$$= \tfrac{1}{2}\rho u^2 + \rho\epsilon u_i + p u_i$$

$$\therefore \quad \underline{\underset{\sim}{q} = \rho(\tfrac{1}{2}u^2 + \epsilon)\underset{\sim}{u} + p\underset{\sim}{u} = \rho(h + \tfrac{1}{2}u^2)\underset{\sim}{u}} \qquad \text{———} \quad (12)$$

$$h = \epsilon + \frac{p}{\rho} = \text{specific enthalpy}$$

<u>Diffusion (coefficients)</u> ie. transport of 'conserved' quantities by particles in a gradient.

<u>Krook's approx</u>

A perturbation $\delta f = f - f_0$ from equilibrium relaxes back exponentially. (Cf Boltzmann's H theorem)

Idea is to linearize the Boltzmann eqn$^{=}$ in δf and try to solve it.

LHS contains the <u>small</u> gradients of f_0, so δf can be ignored

RHS contains δf — analysis of scattering integral depends on the scattering interaction.

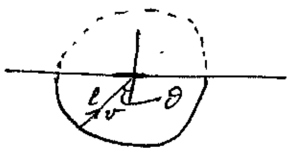
<u>Enskog</u> method (1917)

I shall simply adopt $\underline{C(f) = -\frac{\delta f}{\tau}}$ where $\tau = \frac{\ell}{\bar{v}}$ = collision time

——— (13)

Calculation may look a little formidable. $\exists$ risk of making a mistake.
So better know the answer before we start.

v. useful (essential) in research
(a) for checking
(b) for <u>learning</u> if ans. "wrong".



q is some conserved quantity (per particle)

$$F_q = \frac{n}{4\pi}\int v\cos\theta \underbrace{\left(\ell\cos\theta \frac{-\partial q}{\partial z}\right)}_{\delta q}\sin\theta\, d\theta\, d\phi$$

$$= -\frac{2\pi n v \ell}{4\pi}\int_{-1}^{1}\mu^2 d\mu . \frac{\partial q}{\partial z}$$

$$= -\frac{1}{3} n v \ell \frac{\partial q}{\partial z}$$

If Q is quantity per unit volume

$$\underset{\sim}{F}_Q = -\frac{1}{3} v \ell \nabla Q = -\kappa \nabla Q$$

$$\kappa = \frac{1}{3}\ell v$$

$$\boxed{\frac{\partial Q}{\partial t} - \operatorname{div}(\kappa \nabla Q) = 0}$$

$$\frac{\partial Q}{\partial t} + \operatorname{div}\underset{\sim}{F}_Q = 0$$

Diffusion equation.

This kind of analysis is at the heart of what I am about to do more formally.

<u>Examples</u> (i) radiation. $v = c$ $\quad nq = aT^4$ $\quad F = -\frac{1}{3}c\ell\frac{\partial aT^4}{\partial z} = -4ac\ell T^3\frac{\partial T}{\partial z}$

Define <u>opacity</u> st $\rho\kappa\ell = 1$
κ <u>not</u> to be confused with diffusion coefficient. [I want you to stay awake.]

$$F = -\frac{4acT^3}{3\kappa\rho}\frac{\partial T}{\partial z}$$

This is exact (accidentally) in optically deep limit

(ii) mlt $\quad nq = \rho h' = \rho c_p T'$ $\quad F_c = \left(\frac{1}{3}\right)\rho c_p v \ell'$ $\quad$ Taylor
Prandtl added formula for v

(iii) turbulent ~~[illegible]~~ Reynolds stresses (cf. equation (11) $\rightarrow$ turbulent viscosity) First do molecular

Viscosity

$$\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + F_i\frac{\partial}{\partial p_i}\right)(f_o + \cancel{\delta f}) \simeq -\frac{\delta f}{\tau} = -v'\frac{\delta f}{\ell}$$

ℓ is indep of v for solid spheres, but is v-dependent for charged particles.

$$\delta f \simeq -\ell v^{-1}\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + F_i\frac{\partial}{\partial p_i}\right)f_o \qquad f_o = \frac{n}{(2\pi m kT)^{3/2}}e^{-p^2/2mkT}$$

satisfies $4\pi\int_{-\infty}^{\infty} f_o p^2 dp = n$

Maxwell-Boltzmann

Consider steady flow with no external force

$$\delta f \simeq -\ell v^{-1} v_i \frac{\partial f_o}{\partial x_i} = \frac{m\ell}{kT} v^{-1} v_i v_j f_o \frac{\partial v_j}{\partial x_i}$$

$$\delta\Pi_{ij} = m\int (u_i + v_i')(u_j + v_j')\delta f_o d^3p = \frac{nm^5\ell}{kT(2\pi mkT)^{3/2}}\int v^{-1}(u_i+v_i')(u_j+v_j')(u_k+v_k')(u_\ell+v_\ell')e^{-\frac{mv^2}{2kT}}$$
$$\times\ v^2 dv \frac{\partial u_k}{\partial x_\ell}\sin\theta\, d\theta\, d\phi$$

I could do all 81 integrals (for combinations of i, j, k, ℓ). Alternatively, notice that result (coeff^{nt} of $\frac{\partial u_k}{\partial x_\ell}$) is isotropic $\Rightarrow$ $\delta\Pi_{ij} \propto (\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\frac{\partial u_k}{\partial x_\ell}$

whence

$$\delta\Pi_{ij} = \eta\left(e_{ij} + \tfrac{1}{2}\operatorname{div}\underset{\sim}{u}\,\delta_{ij}\right) \qquad e_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{rate of strain}$$

where

$$\eta = \frac{8\pi m^5 n\ell\alpha}{kT(2\pi mkT)^{3/2}}\int v^5 e^{-mv^2/2kT}dv = 16 n\ell\alpha\sqrt{\frac{2mkT}{\pi}}\int_0^\infty x^5 e^{-x^2}dx$$

where α is a factor that takes angular integrals into account and is independent of i, j, k, ℓ.

The integrals is I_5, where $I_n = \int_0^\infty x^n e^{-x^2}dx$ which satisfies $I_n = \frac{n-1}{2}I_{n-2}$

from which:

$$I_1 = \tfrac{1}{2} \qquad I_0 = \tfrac{\sqrt{\pi}}{2}$$
$$I_3 = \tfrac{1}{2} \qquad I_2 = \tfrac{\sqrt{\pi}}{4}$$
$$I_5 = 1 \qquad I_4 = \tfrac{3\sqrt{\pi}}{8}$$
$$I_7 = 3 \qquad I_6 = \tfrac{15}{16}\sqrt{\pi} \qquad \text{etc}$$

To evaluate α need consider one (simple) nonzero case only. e.g. let $\underset{\sim}{u} = (u(z), 0, 0)$ and consider $\Pi_{12} \propto \int \sin^2\theta\cos^2\phi\sin\theta\, d\theta\, d\phi = \frac{4\pi}{15}$

$$\therefore \quad \underline{\eta = \frac{16}{15}\sqrt{\frac{2mkT}{\pi}}\, n\ell} \qquad \text{Coefficient of shear viscosity}$$

Note that $v'^2 = \frac{3kT}{m} \Rightarrow \quad \eta = \frac{16}{15}\sqrt{\frac{2}{3\pi}}\,\rho v'\ell$

Viscous diffusion coeff (= kinematic viscosity) $= \frac{\eta}{\rho} = \nu$

$$\nu = \frac{16}{15}\sqrt{\frac{2}{3\pi}}\, v'\ell \simeq 0.49 v'\ell$$

NB $\quad c =$ sound speed (adiabatic) $\qquad c^2 = \gamma\frac{p}{\rho} = \frac{5kT}{3m} = \frac{5}{9}v'^2 \qquad \therefore v' = \frac{3}{\sqrt{5}}c$

$$\eta = \frac{16}{15}\sqrt{\frac{6}{5\pi}}\frac{mc}{\sigma} \qquad \text{using} \quad \frac{\rho\ell\sigma}{m} = 1.$$

The numerical factor is for $\sigma =$ const. If σ is velocity dependent, the factor would be somewhat different.

So what is the improvement, if any, of $0.49v'\ell$ over $\frac{1}{3}v'\ell$ from simple analysis. Ans: numerically none, because the formula for the scattering (collision) integral may not be right and in any case it could be scaled by a factor of order unity. But its advantage is that it was derived from a systematic procedure which could be improved by studying more carefully the binary scattering process. It will also facilitate the improvement of the diffusion equation (to be discussed next) for rapid variation (see end of lecture).

Turbulence is often treated in astrophysical fluid dynamics by mixing-length 'theory', which in its most primitive form is based on the kinetic arguments that I have just described for gases. Note that one can use it to transport conserved quantities and just what should be transported is a matter of debate. In Prandtl's original formulation for rectilinear shear flow momentum was transported. Taylor suggested vorticity which appears to work less well in 3-d turbulence, but in layerwise 2-d turbulence (e.g. clear-air turbulence in Earth's atmos) it appears to be batter (leading to 'antifriction') although it is not the whole story.

Thermal conductivity

Work in frame moving with fluid: $\underset{\sim}{u} = \underset{\sim}{0}$.

$$\delta \underset{\sim}{q} = \int \epsilon \underset{\sim}{v} \, \delta f \, d^3p \qquad \delta f \simeq -\tau v_i \frac{\partial f_0}{\partial T}\frac{\partial T}{\partial x_i}$$

$$= -\frac{\ell v_i}{Tv}\left(\frac{mv^2}{2kT} - \frac{3}{2}\right) f_0 \frac{\partial T}{\partial x_i}$$

[I'm assuming for simplicity that $F_i = 0$; if F_i did not vanish one would have a Soret term.]

$$\delta q_i = -\int \tfrac{1}{2} m v^2 v_i \frac{\ell v_j}{vT}\left(\frac{mv^2}{2kT} - \frac{3}{2}\right)\frac{nm^3}{(2\pi mkT)^{3/2}} e^{-mv^2/2kT} v^2 \sin\theta \, dv\, d\theta\, d\phi \frac{\partial T}{\partial x_j}$$

Must have $i = j$ for nonzero integral

$$\therefore \quad \delta q_i = -K\frac{\partial T}{\partial x_i} \qquad \text{where} \quad K = \frac{4nk^2T\ell}{3\sqrt{\pi}\, m}\left(I_7 - \tfrac{3}{2} I_5\right)\sqrt{\frac{m}{2kT}}$$

$$= \frac{\rho k \ell}{m}\sqrt{\frac{2kT}{m}} = \frac{k}{\sigma}\sqrt{\frac{2kT}{\pi m}} \quad \text{thermal conductivity}$$

Estimate of σ for charged particles with charge Ze:

Typical impact parameter a is when $\frac{Z^2e^2}{a} = \frac{1}{2}mv^2$ and $\sigma = \pi a^2 = \frac{4\pi Z^4 e^4}{m^2 v^4}$

For such a velocity dependent $\ell \propto \sigma^{-1} \propto v^4$ integrals should be repeated to get a revised numerical factor

Nb Inverse square law is long range and σ, computed for a pure binary collision, is infinite. Effect of neighbouring charges must be taken into account — screening — although simply truncating scattering integral at the mean distance between neighbouring particles has also been adopted.

Energy equation can now be written

$$\frac{\partial \rho \epsilon}{\partial t} + \frac{\partial}{\partial x_i}\left[\rho\left(h + \tfrac{1}{2}u^2\right)u_i - K\frac{\partial T}{\partial x_i}\right] - \rho \underset{\sim}{u}\cdot\underset{\sim}{\tilde{F}} = 0$$

where I have now replaced $\underset{\sim}{F}/m$ by $\underset{\sim}{\tilde{F}}$ — force per unit mass

In the absence of appreciable motion, $\underset{\sim}{u} = \underset{\sim}{0}$, ρ may be taken to be roughly constant and one gets

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c_p} \nabla(K \nabla T)$$

(if pressure is const. c_p = specific heat at constant pressure)

For this simple discussion take K = const. Then

$$\underline{\frac{\partial T}{\partial t} = \varkappa \nabla^2 T}$$

where $\varkappa = \frac{K}{\rho c_p}$ = thermal diffusion coefficient

This is the diffusion equation

In 1-d it has the elementary solution

$$T = T_0 \frac{1}{2\sqrt{\pi \varkappa t}} e^{-x^2/4\varkappa t}$$

Thus, at $t = 0$, $T = T_0 \, \delta(x)$

δ = delta function

Point source at $t = 0$

$T = 0$ if $x \neq 0$

and for $t > 0$, $T \neq 0$ everywhere. Information travels at infinite speed

Adopting the diffusion of ang. momentum, for example, according to such a ~~problem~~ prescription has led to some inconsistencies in the literature — for example in studying the evolution of accretion discs in which causality was violated. This has led to some banal discussion amongst those who do not understand physics. Evidently, information cannot travel faster than the particles that transports it. To be sure, the particle velocities in the Maxwell-Boltzmann distribution are unbounded, but in an averaged theory such as this one should not expect information to travel appreciably faster than the speed of sound.

The effect of rapid variation on heat diffusion

Repeat the last calculation, but now retain the time derivative in δf:

$$\delta f \simeq -\frac{\ell}{v}\left(\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}\right)f_0 = \frac{-\ell}{Tv}\left(\frac{mv^2}{2kT}-\frac{3}{2}\right)f_0\left(\frac{\partial T}{\partial t}+v_i\frac{\partial T}{\partial x_i}\right)$$

Coefficient of $\frac{\partial T}{\partial t}$ integrates to zero in the integral for q_i, so heat flux is unaltered. But we must now consider the time derivative of the energy. If one is to apply equilibrium thermodynamics to relate ϵ to p, ρ, T the definition must have f_0 in the integral. ∴ there remains in the first term in eqn (9):

$$\int \varepsilon\frac{\partial \delta f}{\partial t}d^3p = -\int \varepsilon\frac{\partial}{\partial t}\left[\frac{\ell}{vT}\left(\frac{mv^2}{2kT}-\frac{3}{2}\right)f_0\left(\frac{\partial T}{\partial t}+v_i\frac{\partial T}{\partial x_i}\right)\right]m^3v^2dv\sin\theta\, d\theta\, d\phi$$

Now it is the term containing $v_i\frac{\partial T}{\partial x_i}$ that integrates to zero, leaving

$$\frac{\partial}{\partial t}\left[\frac{2\pi n m^4\ell}{T(2\pi mkT)^{3/2}}\left(\frac{2kT}{m}\right)^2\left(I_5-\frac{3}{2}I_3\right)\frac{\partial T}{\partial t}\right] = \frac{5}{6\sqrt{\pi}}\frac{\partial}{\partial t}\left(\frac{K}{c^2}\frac{\partial T}{\partial t}\right)$$

K = thermal conductivity

c = sound speed

whence

$$\frac{\partial}{\partial t}(\rho c_p T) + \frac{\partial}{\partial t}\left(\frac{K}{\tilde{c}^2}\frac{\partial T}{\partial t}\right) - \mathrm{div}(K\nabla T) - \rho \underset{\sim}{u}\cdot\underset{\sim}{\tilde{F}} = 0$$

$$\text{where } \tilde{c}^2 = \frac{6\sqrt{\pi}}{5}c^2$$

Equation is now hyperbolic, with propagation speed $\tilde{c}$. It is a kind of 3-d telegraph equation.

The factor $\sqrt{\frac{6\sqrt{\pi}}{5}}$ relating $\tilde{c}$ to c is unreliable, for reasons discussed in relation to viscosity. But, because the nature of the transport is different from that of pressure fluctuations (acoustics) $\tilde{c}$ need not be bounded above by c.