Errors,Chaos and Dynamical Equilibria

Physical and numerical effects of exponential instability in large-N gravitational systems

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Reduction of N-body Problem

In Galactic Dynamics

• **Possible to reduce N-body problem to 3 N one dimensional problems**

(from the point of view of qualitative dynamical properties!)

1- Mean field limit --> particle correlations ignored

--> each particle moves in smooth potential (3 N --> 3+1 d)

2- Assume steady state mean field potential (3 + 1 d --> 3 d)

3- Assume special symmetry (e.g., spherical) (1 d)

Away from idealized situation

Mode Coupling in Large-N Systems

Source	Effect
Asymmetric steady state potential	**Secular evolution** (reasonably well understood)
Time dependent potential	**Violent Relaxation** (of what dynamical origin?)
Discreteness noise	**'Microchaos'** (what is this and what does it do?)

Chaos in Smooth Steady States

(e.g, Henon, Contopoulos, Pfenniger, Merritt, Athanassoula, Quillen, Patsis....60's – 2000's)

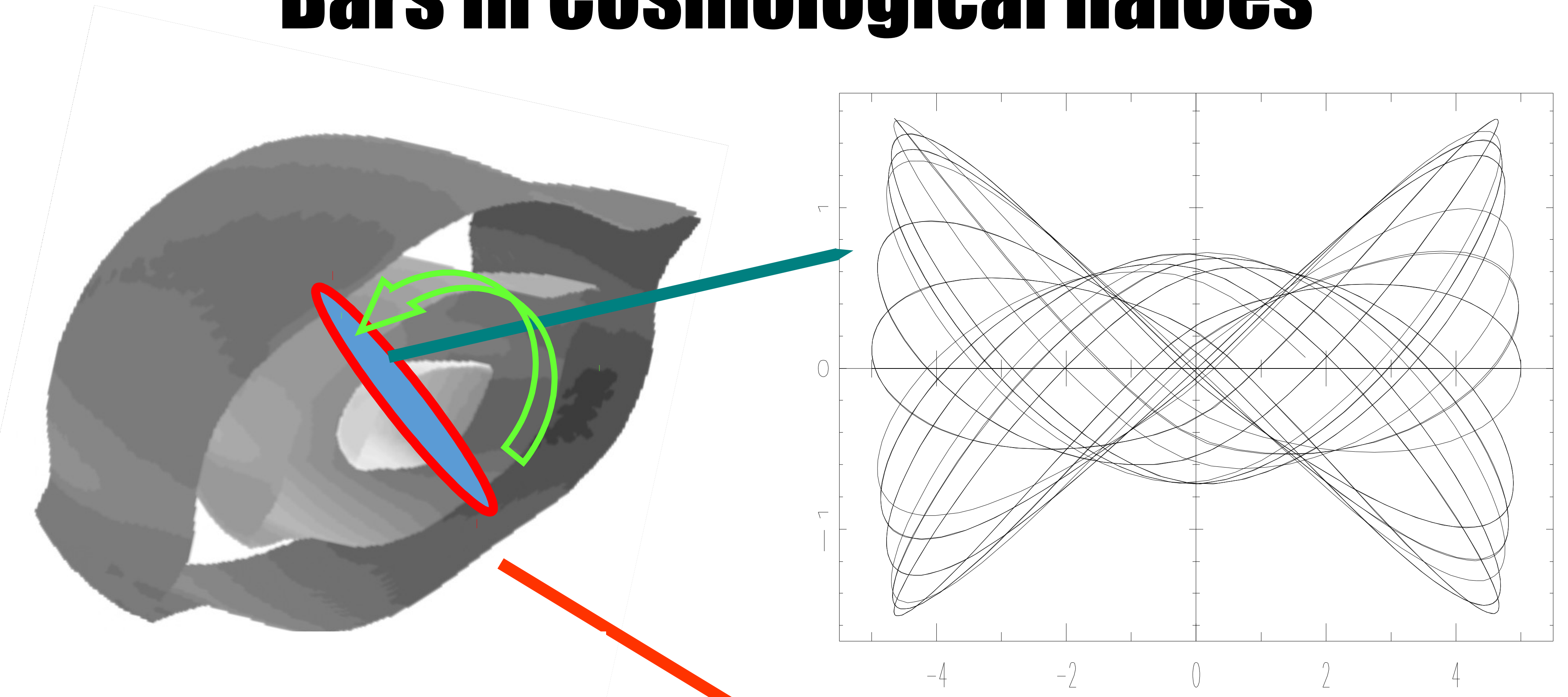
- Requires central concentration (anharmonicty)

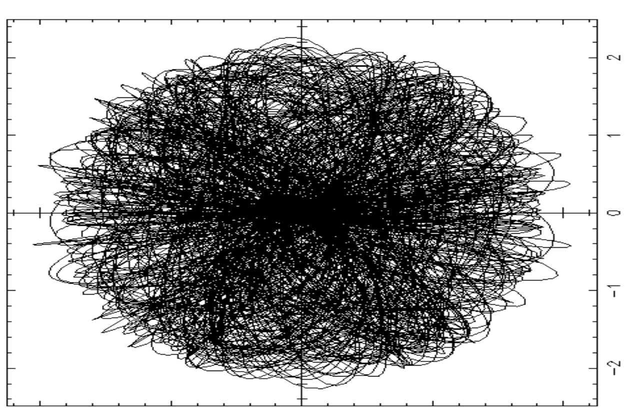
- May be present in asymmetric systems (bars, triaxial ellipticals or haloes...etc)

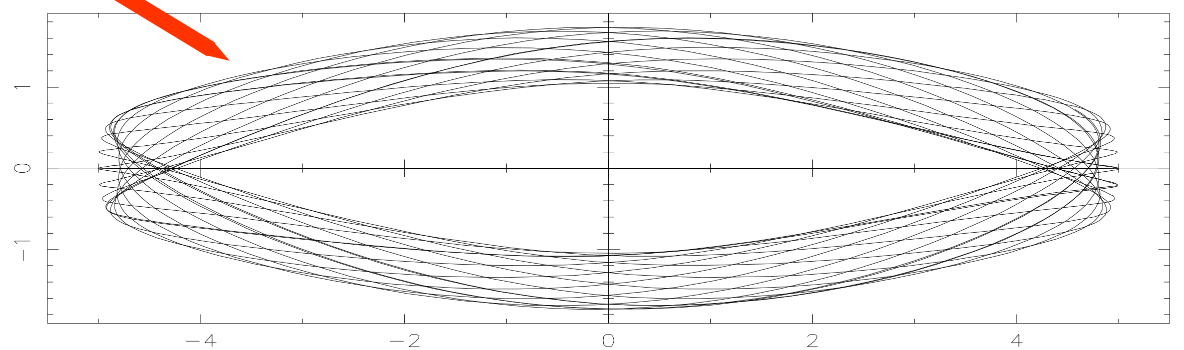
- Strong in systems with multiple non-axisymmetric components – e.g., bar and triaxal halo, double bar etc... **(El-Zant & Shlosman, El-Zant 2002)**

→ **Generally leads to destruction of asymmetry**

Bars in Cosmological Haloes







Confusing Chaology in Full N-body

- Since 1960's known that N-body trajectories exponentially unstable on a dynamical time (Miller)

- Computed trajectories not even reproducible on different machines (Lecar)

- Yet global structure seems relatively unaffected

- But no 'proof' of 'faithfulness' of simulations

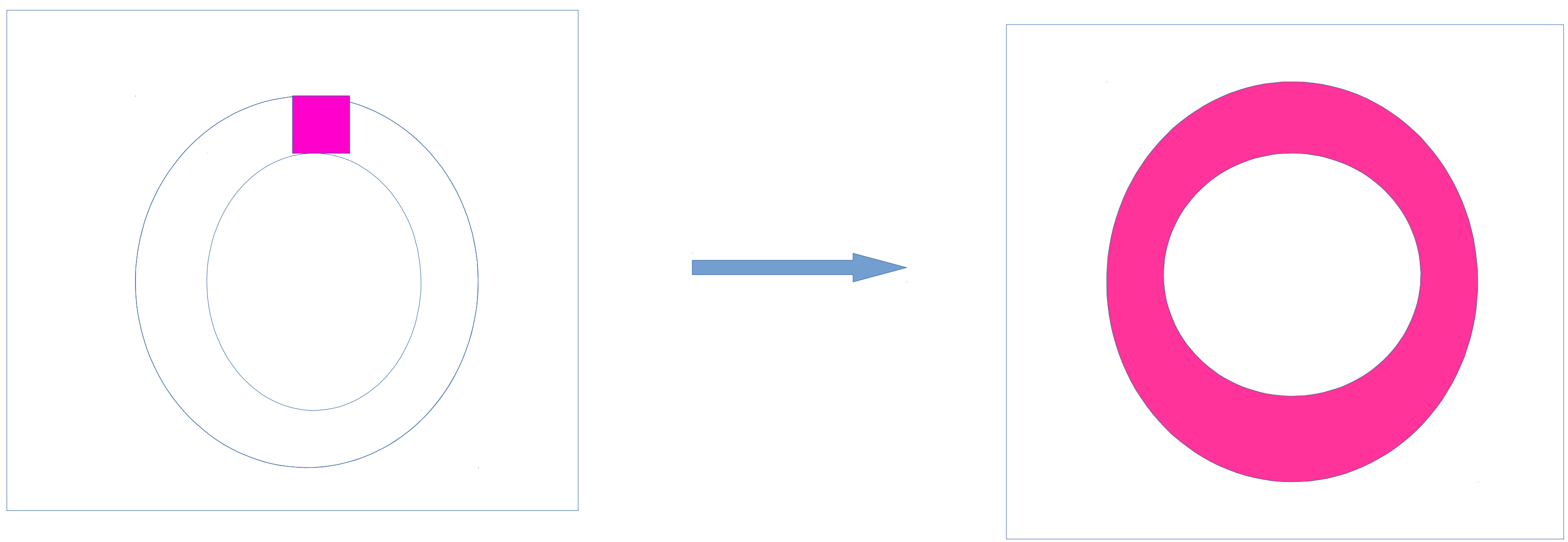
- More confusion (80's), Gurzadyan-Savvidy claim 'collective relaxation' due to exponential instability over short timescales even in spherical steady states

Sourcing the Confusion

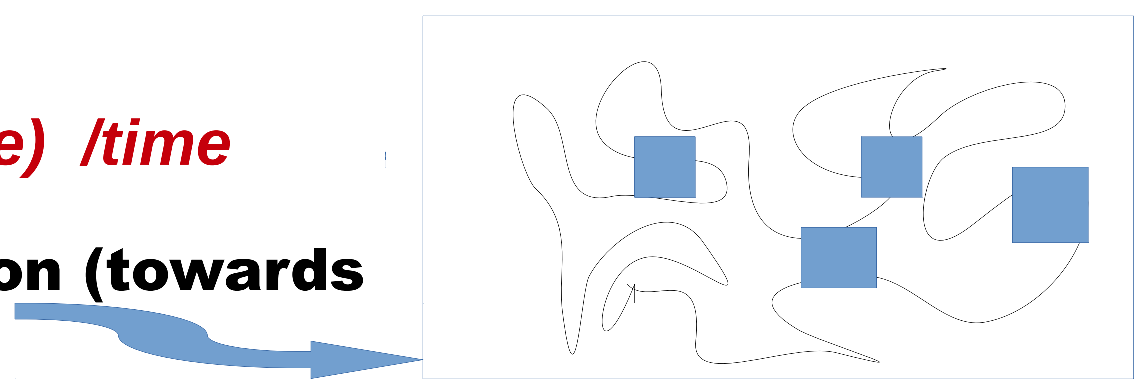
The nature of phase space mixing

Regular – 'constrained'

J = const, constrains motion



Transversal wandering'



Kolmogorov entropy

(at given p-s resolution)

~ ***Log (Coarse grained volume) /time***

Finite! Should imply evolution (towards 'more probable states')!

Partial Resolution (1990's, early 2000's)

Goodman, Heggie, Hut; late Kandrup et al.; Hossendorf and Merritt....

In steady state with integrable background potentials:

(usually toy models with fixed particles!)

1- Exponential Instability persists strictly in linear regime

2- Then divergence scales polynomially (and manly along the motion)

3- Integrals of motion much better conserved and may follow a diffusion scaling (t^1/2) for large t.

4- Trajectories are bound by these integrals and therefore are qualitatively similar to ones in integrable background potential

What about the Theorems?

Kolmogorov, Anosov... (1950's, 60's):

Exponential instability for (almost) all initial conditions should:

1- Positive Kolmogorov entropy

2- Transitive (porous) phase space over chaotic initial conditions

→ **Large scale chaotic mixing and nonconservation of integrals of motion even in steady state over a dynamical time?!**

But what about the mean field limit?

Collisionless Boltzmann Eq. in steady state -->

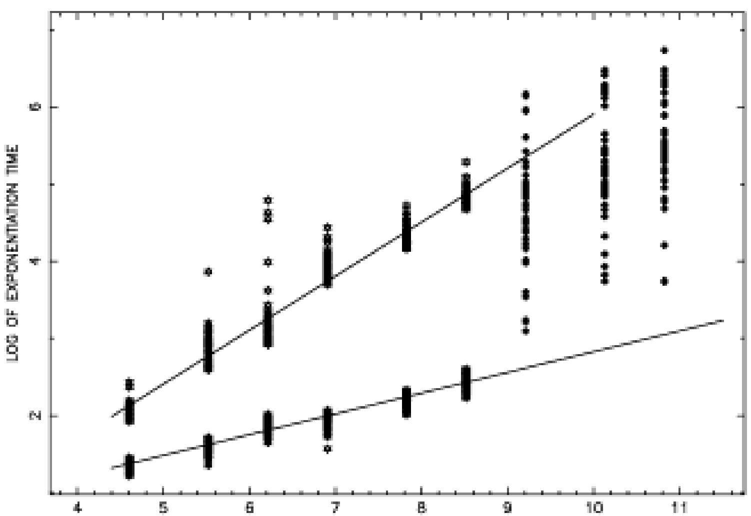
Distribution function of conserved quantities --> orbits are characteristic curves --> no large scale mixing for, e.g., spherical system!

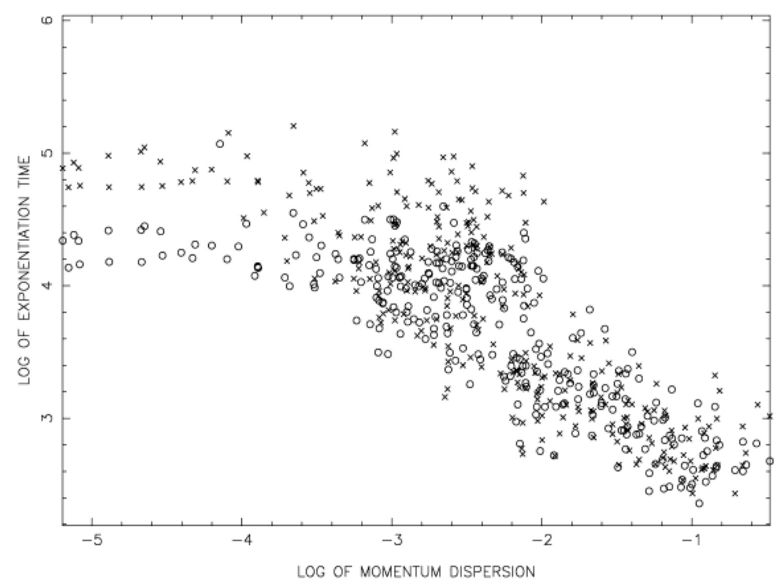
However both the mean feld limit (Braun & Hepp 77) and the 'chaotic mixing theorems' can only be deduced for softened systems (more generally for continuously differentiable systems).

Softened Systems

Exponential Timescales Correlate with Particle number and Conservation of Integrals of Motion

For centrally concentrated systems, a mixed phase space for low N, more regular as N increases





El-Zant (2004)

What about the Errors?

How faithful are our simulations?

Testing reversibility:
** Use high order integrator with small tolerance per timestep (fourth order R-K with 10^-8)
** Run many small N simulations (128, 256, ... 8192) for a few dozen dynamical times
** Average relative RMS error between forward and backward runs over all runs and particles

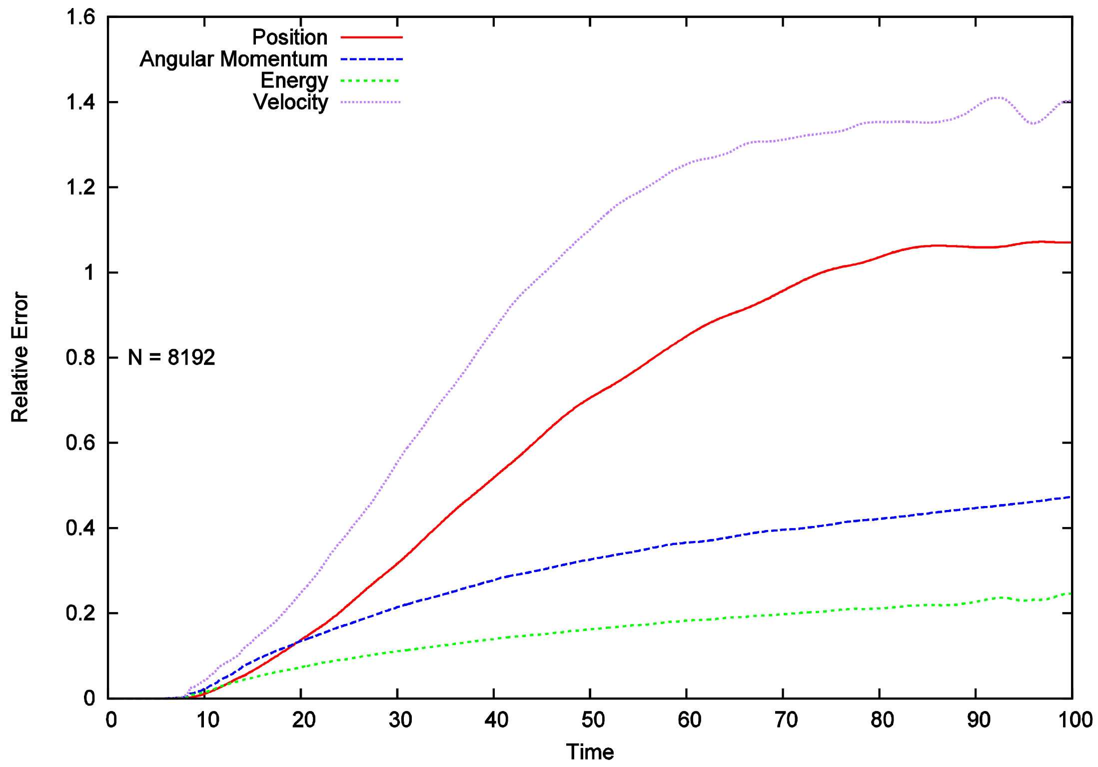
Exponential instability range limited ~ 10 dynamical times

Integrals of motion (of smooth potential) better conserved than dynamical variables

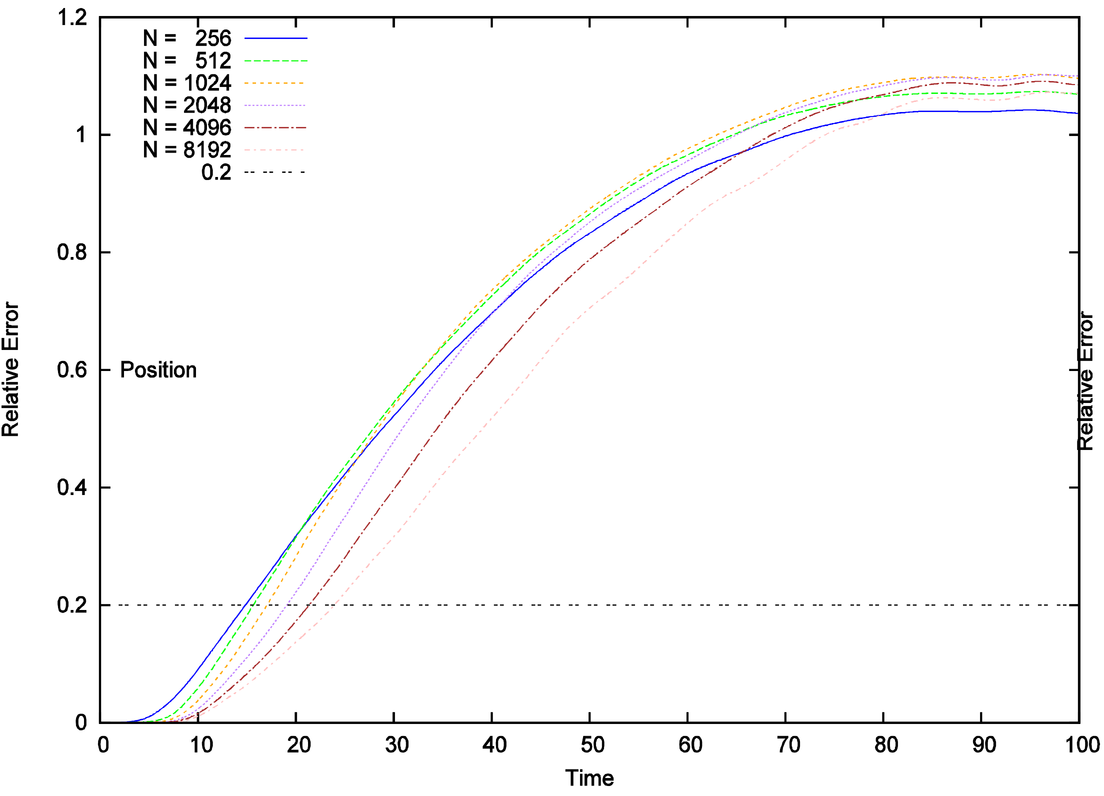
Energy (a scalar) better conserved Than angular momentum

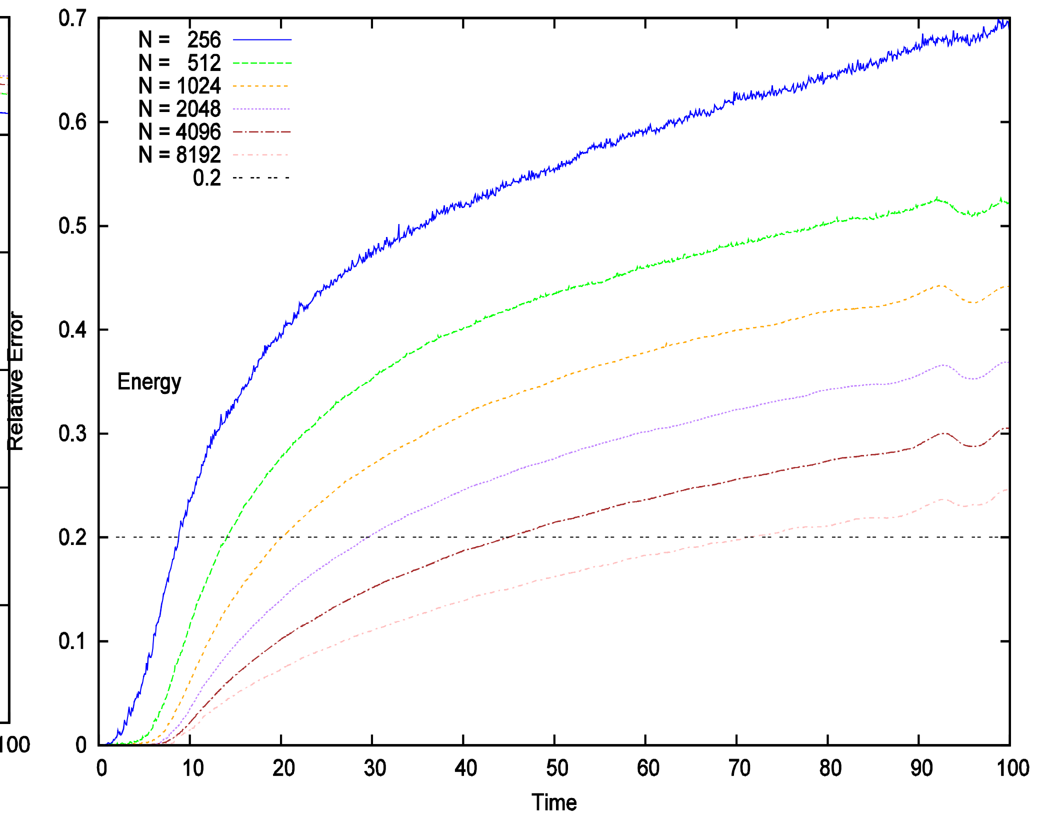
Both reach diffusion limit for N~ 10 000

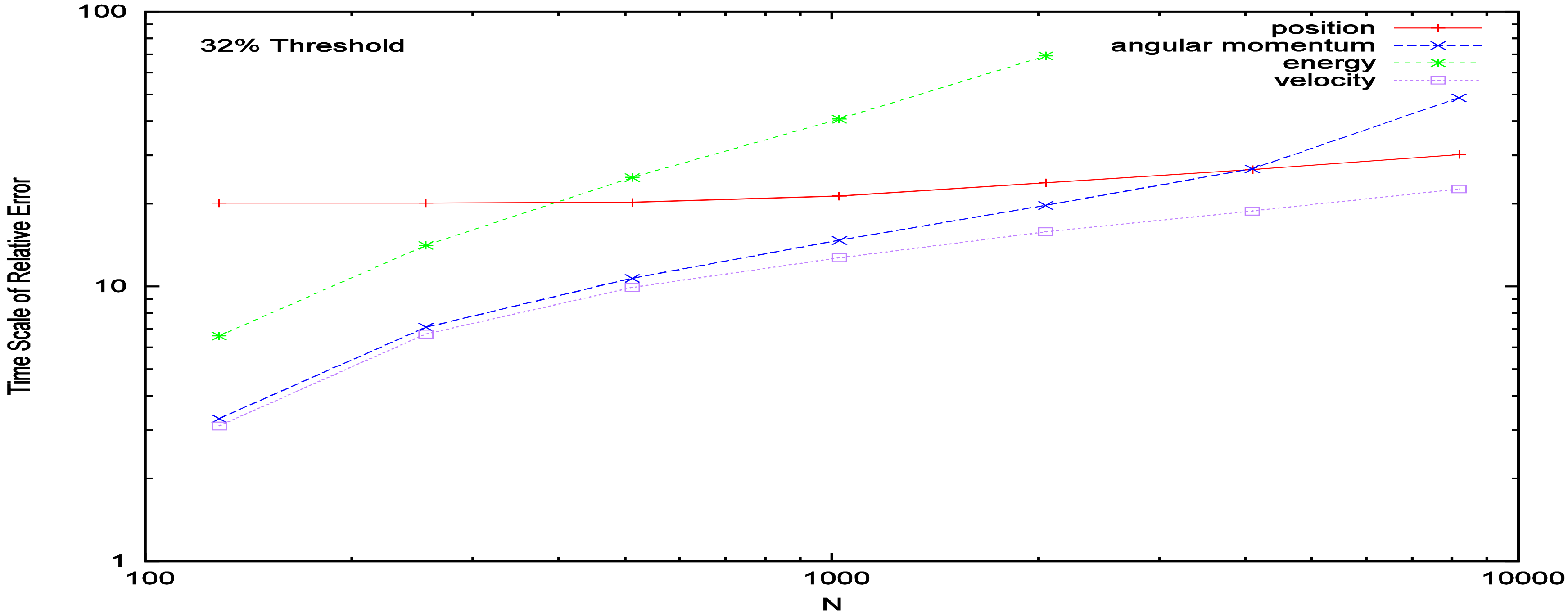
El-Zant, Everitt & Kassem 2014



For, dynamical variables (x,v), post-exponential-stage errors propagate by phase mixing between 'true' and perturbed trajectories
--> 'Ballistic' --> Linear in time
.
Energy errors propagate by means of slower diffusion process
--> Goes as the square root in time
--> as in two body relaxation







So

** 'Microchaos' probably not crucial to macroscopic structure and evolution

** Integrals of motion less affected by errors than dynamical variables
--> modelling systems with distribution function thus determined should be robust

** Angular momentum less well conserved (may affect modelling anisotropies)

** Situation more complicated in asymmetric, clumpy, or time dependent systems

Dynamics on Riemannian Spaces

(an approach to violent relaxation)

$$\delta \int Kinetic\ Energy\, dt = \delta \int ds = 0$$

Maupertuis variational principle --> Lagrange Equations --> Geodesics on 'Lagrangian Manifold'
Riemannian, since the K. E. Is a quadratic form and hence generates local Eucledian structure

The space ('manifold') has dimension equal to the effective number of degrees of freedom...
in the enveloping 3 N d Eucledian space it is

$$ds^2 = W \sum_{3N} (dx^\alpha)^2 . \qquad (1)$$

$$W = E - \Phi = E - \sum_{j>i} \varphi_{i,j}$$

For spherical systems with isotropic velocities information on the stability of motion may be retrieved by studying the scalar curvature.

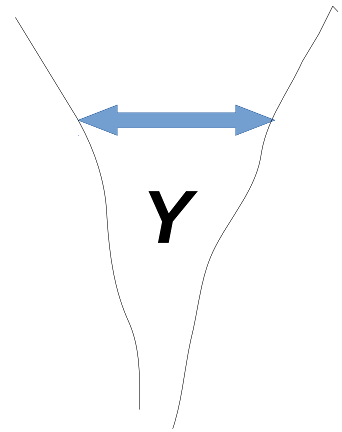
For non-singular (softened) systems:

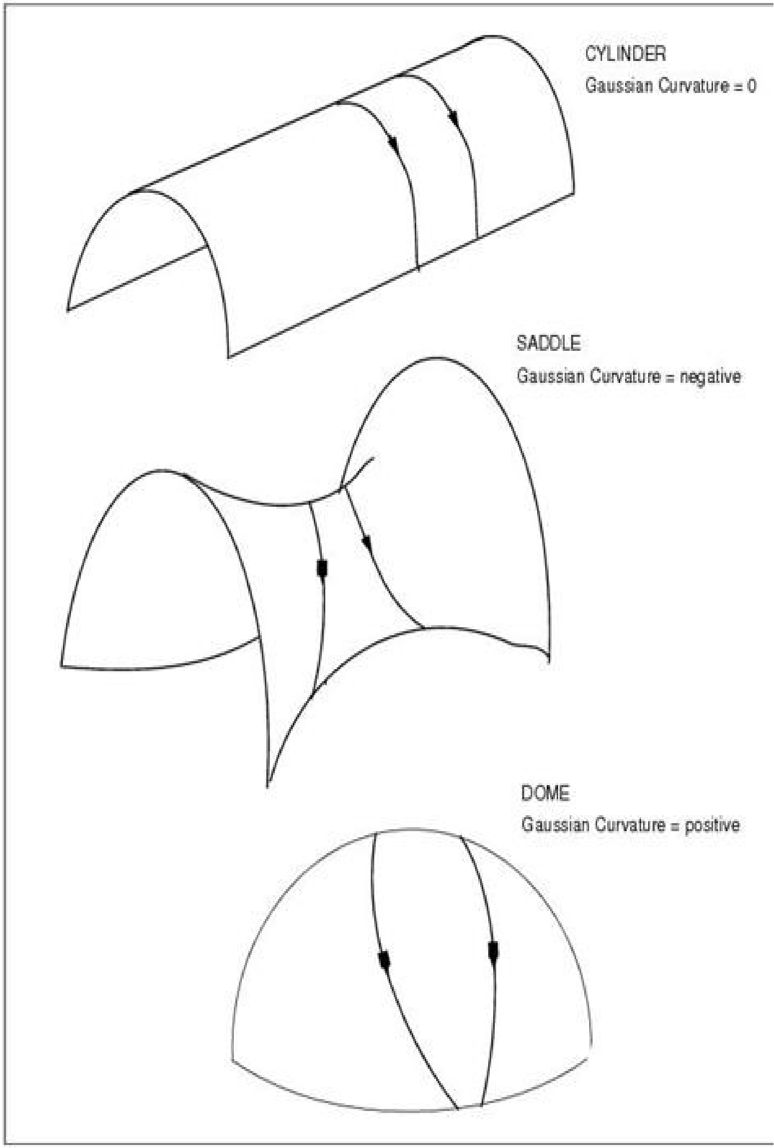
$$R \sim Density \times (Velocity)^2 - (Force)^2$$

$$R = \sum_{u,n} k_{u,n} = -3N \frac{\nabla^2 W}{W^2} - 9N^2 \frac{\| \nabla W \|^2}{4W^3},$$

Curvature and Stability

$$\frac{d^2 Y}{d s^2} \approx -R\,Y$$





Curvature is negative --> Exponential Instability

Curvature is negative in singular potential
~ - Square of Force

Cause of much confusion

From Laminate Modeller Software page

Case of Non-singular potentials

R ~ 0 when 'pressure balances gravity --> dynamical equilibrium

$$W = \frac{1}{2} M \langle v^2 \rangle = \frac{3M}{16\pi G} \frac{\langle a^2 \rangle}{\langle \rho \rangle}.$$

$\langle a^2 \rangle \sim G^2 M^2 / r_s^4$ $\langle \rho \rangle \sim M / r_s^3$

$$\langle v^2 \rangle \sim GM/r_s$$

But not always! e.g., *no cored isotropic isothermal states with R =0*

$$\langle \sigma^2 \rangle_r \int \rho dm = \frac{1}{2} \int \rho \sigma^2 dm. \qquad \text{(A4)}$$

$$\int P dm = 2 \frac{k \langle T \rangle_r}{m_p} \int \rho dm.$$

What does the absence of cores imply?

Formally

$$\frac{d^2 Y}{d t^2} \approx -(\langle R \rangle + Fluctuations) Y$$

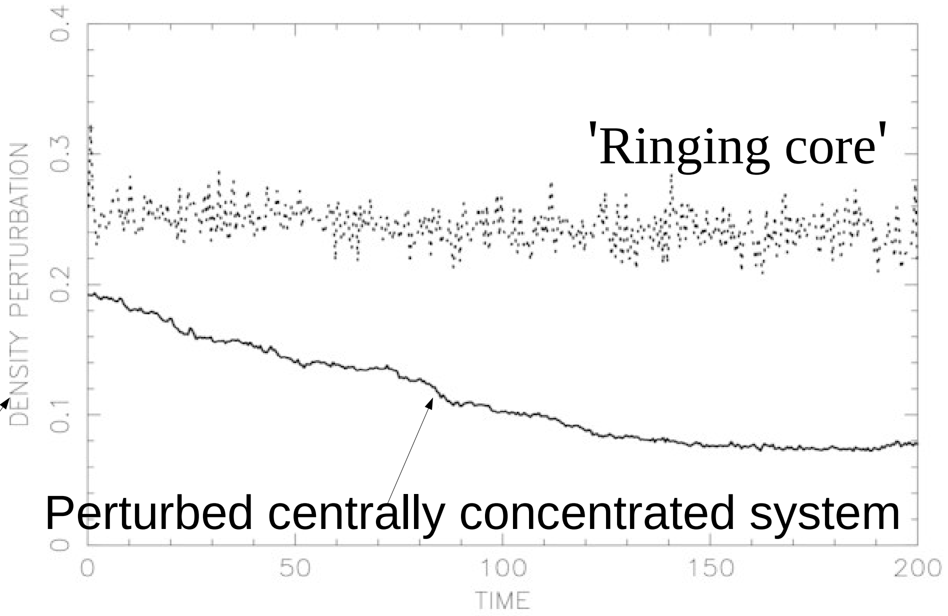
No Fluctuations --> Harmonic Oscillator
With Fluctuations --> Hill's Equation --> instability strips, increasing with level of fluctuation

Intuitively

Systems with cores are nearly harmonic
--> orbital frequency do not depend much on radius
--> tend to 'ring' when perturbed

--> Do not strongly mix
--> Not efficient at washing out perturbations
and reaching steady state

$$\sum_{i=1}^{N} (\rho_i(t=0) - \rho_i(t))^2 / \sum_{i=1}^{N} \rho_i^2(t=0)$$

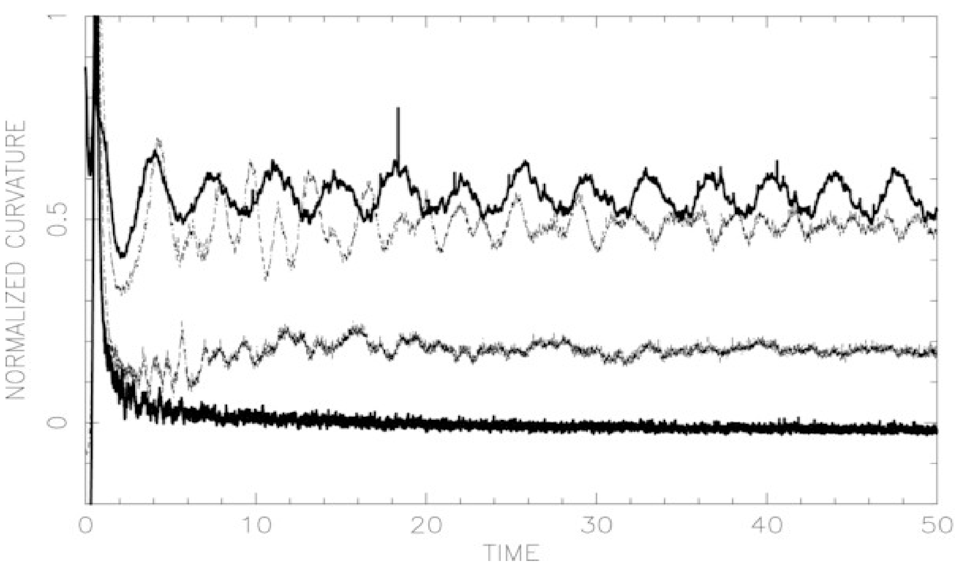


On the final state of V. Relaxation
(of isotropic systems)

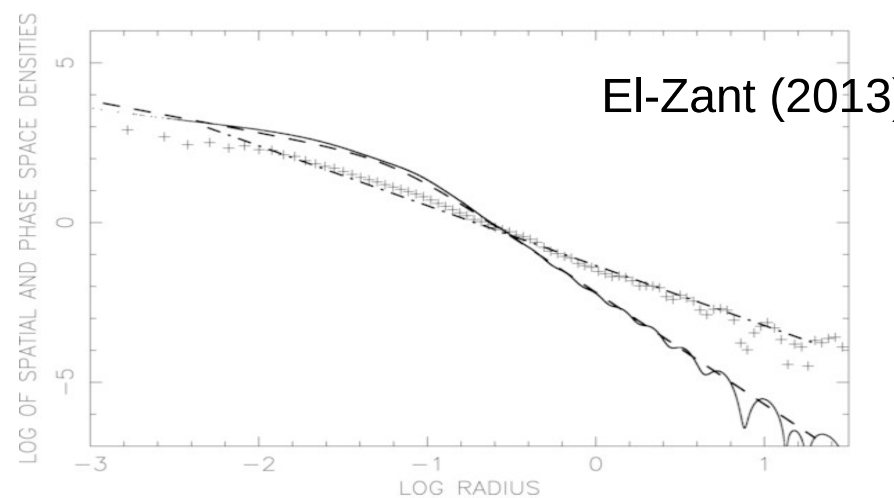
A system starting far from equilibrium will evolve and 'miix' in phase space until orbital divergence destroys collective modes

Collective modes are damped when system finds marginally stable state
--> Efficient in washing away fluctuations

The resulting systems are centrally concentrated and non-isothermal ~ NFW and E gals



Curvature evolution starting with
Virial ratio (top to bottom) 1, 0.5, 0.25, 0.125



Spatial and phase space density profiles and quasi-NFW fits. for equilibrium system with initial Vir =0.125

Summary

- Chaos can be important in centrally concentrated, asymmetrixc systems --> leads to the destruction of very asymmetry causing the chaos --> secular evolution
- 'Microchaos' in N-body systems mainly a manifestation of improper linearization --> probably not physically important
- Nevertheless, trajectory divergence leads to complete loss of information of dynamical variables of computed trajectories of ~ few 10 d. times
- Collisionless limit remains valid, in the sense integrals of motion (when they exist) are relatively well conserved, and their distribution unaffected.
- Suggest that bulk properties inferred from simulations are robust, but some less than others – e.g., direction dependent quantities such as anisotropies.
- Chaos in violent relaxation is 'self defeating', the resultant 'mixing' destroying the collective modes that cause violent relaxation in first place.
- A system therefore may not end in a 'most probable state', but when it finds one that is efficient at damping collective modes --> 'a 'marginally stable state' from the point of view of orbital divergence