ISIMA report on The elliptical Hill's problem and its application to escape from star clusters

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Abstract

The dynamics of escape from a cluster which moves on a circular orbit in the Galaxy can be studied in the context of Hill's equation. Surprisingly, analytical studies of the escape of stars from star clusters beyond this traditional formulation are rare. In this study, we generalized Hill's formalism for a cluster moving on an elliptical orbit. Using this more generalized approach, we calculated the time-dependent escape rate for the circular case. This approach can be used to calculate the escape rate for the elliptical case in the future.

1. Introduction

As a star cluster orbits around the center of the galaxy, the tidal forces induced by the galactic potential, create unstable orbits on which stars can escape the cluster. In the simplified case, where the cluster move on a circular orbit around the galactic center, and we treat the galaxy and cluster as point masses, the system: star, cluster and galaxy can be treated as restricted three body problem known as the "Hill's problem" (Heggie 2001). G.W Hill studied the effect of the tidal force of the sum on the moon's orbit around the earth. In contrast to Hill's problem the potentials of the cluster and the galaxy are not Keplerian, nevertheless this is a good starting point to study this problem (Heggie 2001). Fukushige & Heggie (2000) studied the escape rate from a cluster moving on circular orbit for a more physically motivated cluster and Galaxy potentials and their results proved to be of importance in the interpretation of N-body simulations (Baumgardt 2001). Surprisingly, this generalized form of Hill's problem and its application to the study of a star cluster moving on an elliptic orbit are still only partially understood. However, using N-body simulations Baumgardt & Makino (2003) showed that the eccentricity of the cluster's orbit play an important role in the cluster dynamics. This motivated us to study the generalized Hill's problem and its application to the escape from star clusters. In section 2 we derive the equation of motion for a general spherical galactic and cluster potentials and show that the Lagrangian points can be generalized under some constraints on the potentials. In section 3 we obtain the linearized equation of motion of a star near the Lagrangian point. In section 4 we use the linearized equation to calculate the time dependent escape rate for the circular case. We summarize in section 5 and give an outline for a future study.

2. Generalized Hill's problem

In this section we derive the equation of motion for a test star in a star-cluster with a spherical potential $\Phi_c\left(r\right)$, orbiting galaxy with a spherical potential $\Phi_g\left(R\right)$.

Let $\mathbf{r} = (x, y, z)$ be the coordinates of the test star in a rotating frame centered at the cluster center which is located at the position $\mathbf{R} = R\hat{\mathbf{R}}$ relative to the galactic center (see Figure 1). The motion of the cluster is given by $\ddot{\mathbf{R}} = -\nabla_{\mathbf{R}}\Phi_g(R)$ or

$$\Omega^2 = \dot{\phi}^2 = \frac{\ddot{R}}{R} + \frac{1}{R}\Phi_g'(R) \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{L} = R\left(2\dot{R}\boldsymbol{\Omega} + R\dot{\boldsymbol{\Omega}}\right) = 0 \tag{2}$$

where $\Omega = \dot{\phi}$ is the angular velocity and $\mathbf{L} = R^2\boldsymbol{\Omega} = R^2\Omega\hat{\mathbf{z}}$ is the angular momentum.

The motion of the test star is given by

$$\frac{d(\mathbf{R}+\mathbf{r})}{dt} = \boldsymbol{\Omega}\times(\mathbf{R}+\mathbf{r}) + \dot{\mathbf{R}} + \dot{\mathbf{r}} \tag{3}$$

where $\dot{\mathbf{X}}$ represents time derivative of the vector $\mathbf{X}$ in the rotating frame, that is $\dot{\mathbf{R}} = \dot{R}\hat{\mathbf{R}}$ and $\dot{\mathbf{r}} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}}$.

The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\left|\boldsymbol{\Omega}\times(\mathbf{r}+\mathbf{R}) + \dot{\mathbf{r}} + \dot{\mathbf{R}}\right|^2 - \Phi_g(|\mathbf{R}+\mathbf{r}|) - \Phi_c(\mathbf{r}), \tag{4}$$

where the momentum is

$$\mathbf{p} = \nabla_{\dot{r}}\mathcal{L} = \boldsymbol{\Omega}\times(\mathbf{r}+\mathbf{R}) + \dot{\mathbf{r}} + \dot{\mathbf{R}}, \tag{5}$$

and the Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}p^2 - \left(\boldsymbol{\Omega}\times(\mathbf{r}+\mathbf{R}) + \dot{\mathbf{R}}\right)\cdot\mathbf{p} + \Phi_g(|\mathbf{R}+\mathbf{r}|) + \Phi_c(r). \tag{6}$$

Assuming $r \ll R$ we can expand the galactic potential around R.

$$\Phi_g(|\mathbf{R}+\mathbf{r}|) = \Phi_g(R) + \frac{\Phi_g'(R)}{R}\left(\frac{r^2}{2} + Rx\right) + \frac{1}{2}\left(\Phi_g''(R) - \frac{\Phi_g'(R)}{R}\right)x^2 \tag{7}$$

2.1. Equation of motion

The equation of motion can be read directly from the Hamiltonian

$$\dot{\mathbf{p}} = -\nabla_{\mathbf{r}}\mathcal{H} = \mathbf{p}\times\boldsymbol{\Omega} - \Phi_g'(|\mathbf{R}+\mathbf{r}|)\frac{\mathbf{R}+\mathbf{r}}{|\mathbf{R}+\mathbf{r}|} - \Phi_c'(r)\frac{\mathbf{r}}{r} \tag{8}$$

and

$$\ddot{\mathbf{r}} = -2\boldsymbol{\Omega}\times\dot{\mathbf{r}} + 2\frac{\dot{R}}{R}\boldsymbol{\Omega}\times\mathbf{r} - \nabla_{\mathbf{r}}\Phi_{eff} \tag{9}$$

where the effective potential is given by

$$\begin{aligned}\Phi_{eff}(\mathbf{r}) &= -\frac{1}{2}\Omega^2 r^2 + \frac{1}{2}\Omega^2 z^2 - \Phi_g'(R)\,x + \Phi_g(|\mathbf{R}+\mathbf{r}|) + \Phi_c(r) - \Phi_g(R) \\ &= \frac{1}{2}\Omega^2 z^2 + \Phi_c(r) - \frac{1}{2}\frac{\ddot{R}}{R}\left(r^2 - x^2\right) + \frac{1}{2}\left(\kappa^2 - 4\Omega^2\right)x^2 \end{aligned} \tag{10}$$

and we defined

$$\kappa^2 = \Phi_g''(R) + 3\Omega^2, \tag{11}$$

and used the equations of motion of the cluster (equations (1) and (2)). Note that κ is *not* the epicyclic frequency at radius R, as the value of Ω is not the angular velocity of a *circular* orbit of radius R.

Therefore, the equations of motions are

$$\ddot{x} = -2\frac{\dot{R}}{R}\Omega y - \left(\kappa^2 - 4\Omega^2\right)x + 2\Omega\dot{y} - \Phi_c'(r)\frac{x}{r} \tag{12}$$

$$\ddot{y} = 2\frac{\dot{R}}{R}\Omega x - 2\Omega\dot{x} + \frac{\ddot{R}}{R}y - \Phi_c'(r)\frac{y}{r} \tag{13}$$

$$\ddot{z} = -\Omega^2 z + \frac{\ddot{R}}{R}z - \Phi_c'(r)\frac{z}{r} \tag{14}$$

2.2. Zero-Velocity Surfaces

In the circular case the energy is given by

$$E = \mathcal{H} = \frac{1}{2}\dot{r}^2 + \Phi_{eff}(\mathbf{r}), \tag{15}$$

is an integral of motion (The Jacobi integral). Therefore, the stars are confined to the phase-space volume enclosed within the zero velocity surface $E = \Phi_{eff}(\mathbf{r})$, known as the "Hill's curve" in two-dimensional problem (see Figure 2). The Lagrange points are the two equilibrium points given by $\mathbf{r}(t) = (\pm r_L, 0, 0)$ where r_L is constant in time.

In the elliptical case E is not an integral of motion. However, in some cases we can generalized the Lagrange points by an orbit of the form $\mathbf{r}_L(t) = (\pm r_L(t), 0, 0)$. For such an orbit, using the equations of motions, we have

$$\ddot{x} = -\left(\kappa^2 - 4\Omega^2\right)x - \Phi_c'(|x|)\frac{x}{|x|}, \tag{16}$$

$$\ddot{y} = \frac{2\Omega}{R}\left(\dot{R}x - R\dot{x}\right) = 0. \tag{17}$$

From equation (17) we obtain $r_L = \alpha R$ where α is constant in time, and thus using equation (16) we have $\ddot{r}_L = r_L\ddot{R}/R = \alpha\ddot{R}$, where α is the solution of

$$\Phi_c'(\alpha R) = -\alpha\ddot{R} - (\kappa^2 - 4\Omega^2)\alpha R = \alpha\Phi_g'(R) - \alpha R\Phi_g''(R). \tag{18}$$

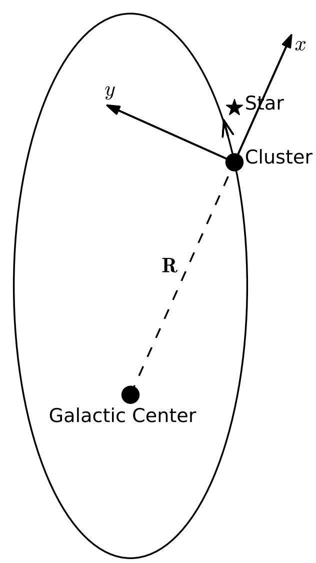

Fig. 1.— A star in a cluster rotating around the galactic center in an elliptical orbit

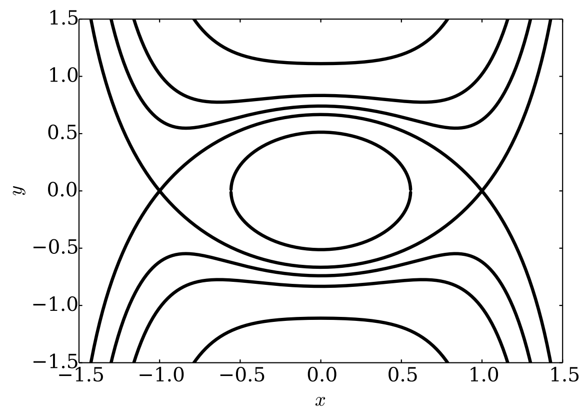

Fig. 2.— Hill's curves

If such a solution exists

$$r_L = -\Phi_c'(r_L)/\left(\kappa^2 - 4\Omega^2 + \ddot{R}/R\right), \tag{19}$$

and the critical energy is given by

$$\begin{aligned} E_c &= \frac{1}{2}\dot{r}_L^2 + \Phi_{eff}(\mathbf{r}_L) = \Phi_c(r_L) - \frac{1}{2}\Phi_c'(r_L)\, r_L + \frac{1}{2}\dot{r}_L^2 - \frac{1}{2}\frac{\ddot{R}}{R}r_L^2 \\ &= \frac{3}{2}\Phi_c(r_L)\left(1 - \frac{1}{3}\frac{d\log[r\Phi_c(r)]}{d\log r}\bigg|_{r=r_L}\right) + \frac{1}{2}\left(\frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R}\right)r_L^2. \end{aligned} \tag{20}$$

2.3. Regularization

We define a regularized position vector $\tilde{\mathbf{r}} = \mathbf{r}/(\alpha R)$ and replace the time by a phase $\tau(t) = \int_0^t \Omega dt$ (recall that $\Omega = \dot{\phi}$). Using the notation $\tilde{\mathbf{r}}' = \partial\tilde{\mathbf{r}}/\partial\tau$ we have

$$\tilde{\mathbf{r}}' = \frac{1}{\alpha\Omega}\left(\frac{\dot{\mathbf{r}}}{R} - \frac{\mathbf{r}\dot{R}}{R^2}\right). \tag{21}$$

The equations of motion, after some simplification and using the fact that $L = R^2\Omega$ is constant, are

$$\tilde{\mathbf{r}}'' = \frac{1}{\alpha}\frac{1}{\Omega^2}\left(\frac{\ddot{R}}{R} + \kappa^2\frac{\ddot{\mathbf{r}}}{R} - \frac{\mathbf{r}\ddot{R}}{R^2}\right) = -\nabla_{\tilde{\mathbf{r}}}\tilde{\Phi}_{eff}(\tilde{\mathbf{r}}) - 2\hat{\mathbf{z}}\times\tilde{\mathbf{r}}', \tag{22}$$

where the effective potential is given by

$$\tilde{\Phi}_{eff}(\tilde{\mathbf{r}}) = \frac{\Phi_{eff}(\mathbf{r})}{\Omega^2\alpha^2R^2} = \frac{1}{2}\tilde{z}^2 + \frac{1}{2}\beta\tilde{x}^2 + \tilde{\Phi}_c(\tilde{r}), \tag{23}$$

where

$$\tilde{\Phi}_c(\tilde{r}) = \frac{\Phi_c(\tilde{r}\alpha R)}{\alpha^2\Omega^2R^2} = \frac{1}{r_L^2\Omega^2}\Phi_c(\tilde{r}r_L), \tag{24}$$

and

$$\beta(R) = \frac{1}{\Omega^2}\left(\frac{\ddot{R}}{R} + \kappa^2 - 4\Omega^2\right) = -\Omega^{-2}\Phi_c'(r_L)/r_L, \tag{25}$$

where we used equation (18).

The regularized "energy" is given by

$$\tilde{E} = \frac{1}{2}\tilde{\mathbf{r}}'^2 + \tilde{\Phi}_{eff}(\tilde{\mathbf{r}}), \tag{26}$$

and the Lagrange points are now stationary and are given by $\tilde{\mathbf{r}} = \pm\hat{\mathbf{x}}$. The phase dependent critical energy is given by

$$\tilde{E}_c = \frac{1}{2}\beta + \tilde{\Phi}_c(1) = \frac{3}{2}\Phi_c(1)\left(1 - \frac{1}{3}\frac{d\log[\tilde{r}\Phi_c(\tilde{r})]}{d\log\tilde{r}}\bigg|_{r=r_L}\right). \tag{27}$$

The phase dependent Hamiltonian is given by

$$\begin{aligned}\tilde{\mathcal{H}} &= \frac{1}{2}\tilde{\mathbf{r}}'^2 + \tilde{\Phi}_{eff}\left(\tilde{\mathbf{r}}\right) \\ &= \frac{1}{2}\tilde{p}^2 - \tilde{\mathbf{p}}\cdot\left(\hat{\mathbf{z}}\times\tilde{\mathbf{r}}\right) + \frac{1}{2}\tilde{\mathbf{r}}^2 + \Phi_{eff}\left(\tilde{\mathbf{r}}\right), \qquad (28)\end{aligned}$$

where the conjugate momentum is given by

$$\tilde{\mathbf{p}} = \tilde{\mathbf{r}}' + \hat{\mathbf{z}}\times\tilde{\mathbf{r}}. \qquad (29)$$

2.4. Keplerian potential

It is customary to assume that the potential of the cluster is Keplerian $\Phi_c\left(r\right) = -GM_c/r$. In this case, by equation (18), a generalized Lagrange orbit exist if the potential of galaxy is of the from

$$\Phi_g\left(R\right) = -\frac{1}{3}\frac{GM_c}{\alpha^3 R} + \frac{1}{2}c_1 R^2 + c_2, \qquad (30)$$

where c_1 and c_2 are constants in time.

Assuming the Galactic potential is also Keplerian $\Phi_g\left(R\right) = -GM_g/R$ we have $\alpha^3 = M_c/\left(3M_g\right)$ and

$$r_L^3 = \frac{1}{3}\frac{M_c}{M_g}R^3. \qquad (31)$$

In this case,

$$\beta = -\frac{GM_c}{\Omega^2 r_L^3} = -\frac{3GM_g}{\Omega^2 R^3}$$

and from the equations of Keplerian motion we deduce that $\beta = -3(1+e\cos\phi)^{-1}$, where e is the eccentricity and ϕ is measured from perigalacticon. Thus, the regularized potential is given by

$$\tilde{\Phi}_{eff} = \frac{1}{2}\left[\tilde{z}^2 - 3(1+e\cos\phi)^{-1}\left(\tilde{x}^2 + \frac{2}{\tilde{r}}\right)\right] \qquad (32)$$

and the equations of motion are

$$\begin{aligned}\tilde{x}'' &= 2\tilde{y}' + 3(1+e\cos\phi)^{-1}\left(1-\frac{1}{\tilde{r}^3}\right)\tilde{x}, \qquad (33)\\ \tilde{y}'' &= -2\tilde{x}' - 3(1+e\cos\phi)^{-1}\frac{\tilde{y}}{\tilde{r}^3}, \qquad (34)\\ \tilde{z}'' &= -\tilde{z} - 3(1+e\cos\phi)^{-1}\frac{\tilde{z}}{\tilde{r}^3}. \qquad (35)\end{aligned}$$

3. The linearized Hill's equations

Let consider the two dimensional problem ($\tilde{z} = 0$) assuming that the cluster and the Galactic potentials are Keplerian. By defining $\tilde{x} = 1 + \xi$, the potential near the Lagrange point (up to second order) is given by

$$\tilde{\Phi}_{eff} = \left(1 - \frac{1}{3}\tilde{y}^2 + \xi^2\right)\tilde{E}_c\left(\phi\right), \qquad (36)$$

where $\tilde{E}_c(\phi)$ is the phase dependent critical energy

$$\tilde{E}_c = -\frac{9}{2}\left(1 + e\cos\phi\right)^{-1}, \tag{37}$$

and the energy of a star is given by

$$\tilde{E} = \frac{1}{2}\left(\xi'^2 + \tilde{y}'^2\right) + \tilde{\Phi}_{eff} \tag{38}$$

3.1. Linearized equation for the circular case

In the circular case the linear approximation is given by

$$\tilde{\phi}_{eff} = \frac{3}{2}\tilde{y}^2 - \frac{9}{2}\xi^2 - \frac{9}{2} \tag{39}$$

and the energy is given by

$$\tilde{E} = \frac{1}{2}\tilde{r}'^2 - \frac{9}{2}\left(1 - \frac{1}{3}\tilde{y}^2 + \xi^2\right). \tag{40}$$

The linear equation of motion are

$$\xi'' = 9\xi + 2y', \tag{41}$$

$$y'' = -3\xi - 2\xi'. \tag{42}$$

Let $\mathbf{v} = (\xi, \tilde{y}, \xi', \tilde{y}')$, the equation of motion are given by

$$\mathbf{v}' = A\mathbf{v}, \tag{43}$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9 & 0 & 0 & 2 \\ 0 & -3 & -2 & 0 \end{pmatrix}. \tag{44}$$

Let Q be a matrix such that $D = Q^{-1}AQ$ is diagonal. Then

$$Q^{-1}\mathbf{v}' = DQ^{-1}\mathbf{v} \tag{45}$$

and

$$\mathbf{v}(t) = Qe^{D\tau}Q^{-1}\mathbf{v}_0. \tag{46}$$

The real eigenvalues of A are

$$\lambda_{1,2} = \pm\sqrt{1 + 2\sqrt{7}}, \tag{47}$$

and the imaginary are

$$\lambda_{3,4} = \pm i\sqrt{2\sqrt{7} - 1}. \tag{48}$$

3.2. Linearized equation for the elliptical case

Given the vector $\mathbf{v}=(\xi,\tilde{y},\xi',\tilde{y}')$ the equation of motion are given by

$$\mathbf{v}'=A\mathbf{v} \tag{49}$$

where

$$A=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \frac{9}{(1+e\cos(\phi_0+\tau))} & 0 & 0 & 2\\ 0 & -\frac{3}{(1+e\cos(\phi_0+\tau))} & -2 & 0 \end{pmatrix}, \tag{50}$$

and we have written $\phi=\phi_0+\tau$, where ϕ_0 is a constant.

These equation of motion cannot be solved analytically, as we did for the circular case, because of the explicit phase dependency. However, assuming small eccentricity $e\ll 1$ an analytical solution can be obtains perturbatively. First we approximate equation (49) by

$$\mathbf{v}'=A_0\mathbf{v}-\left[e\cos\left(\phi_0+\tau\right)-e^2\cos^2\left(\phi_0+\tau\right)\right]A_1\mathbf{v}, \tag{51}$$

where

$$A_0=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 9 & 0 & 0 & 2\\ 0 & -3 & -2 & 0 \end{pmatrix};\ A_1=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 9 & 0 & 0 & 0\\ 0 & -3 & 0 & 0 \end{pmatrix}. \tag{52}$$

Since A_0 is diagonalizable there exists a non-singular matrix Q such that $D=Q^{-1}A_0Q$ is diagonal. Setting $\mathbf{u}=Q^{-1}\mathbf{v}$ we have

$$\mathbf{u}'=D\mathbf{u}-e\cos\left(\phi_0+\tau\right)Q^{-1}A_1Q\mathbf{u}, \tag{53}$$

to first order in e.

Now let's expand $\mathbf{u}$ with respect to the small parameter e, up to first order:

$$\mathbf{u}\left(\tau\right) = \mathbf{u}_0\left(\tau\right)+e\mathbf{u}_1\left(\tau\right). \tag{54}$$

Plugging this back to the equation of motion we obtain

$$\mathbf{u}_0'\left(\tau\right) = D\mathbf{u}_0\left(\tau\right), \tag{55}$$

$$\mathbf{u}_1'\left(\tau\right) = A_0\mathbf{u}_1\left(\tau\right)-\cos\left(\phi_0+\tau\right)Q^{-1}A_1Q\mathbf{u}_0\left(\tau\right), \tag{56}$$

Therefore

$$\mathbf{u}_0\left(\tau\right) = e^{D\tau}\mathbf{u}_0\left(0\right), \tag{57}$$

$$\mathbf{u}_1\left(\tau\right) = e^{D\tau}\left[\mathbf{u}_1\left(0\right)-P_1\left(\tau\right)\mathbf{u}_0\left(0\right)\right], \tag{58}$$

where

$$P_1\left(\tau\right) = \int_0^{\tau}\cos\left(\phi_0+s\right)e^{-Ds}Q^{-1}A_1Qe^{Ds}ds. \tag{59}$$

Therefore, up to first order in e

$$\mathbf{u}(\tau) = e^{D\tau} \{1 - eP_1(\tau)\} \mathbf{u}(0), \tag{60}$$

and

$$\mathbf{v}(\tau) = Qe^{D\tau}Q^{-1} \left\{1 - eQP_1(\tau) Q^{-1}\right\} \mathbf{v}(0). \tag{61}$$

In the next section we shall be paying particular attention to the evolution of ξ, and thus we can write

$$\xi(\tau) = \mathbf{k}(\tau) \cdot \mathbf{v}(0),$$

where

$$\mathbf{k}(\tau) \quad = \quad \hat{\mathbf{x}}^T Qe^{D\tau} (1 - eP_1(\tau)) Q^{-1}, \tag{62}$$

and $\hat{\mathbf{x}} = (1, 0, 0, 0)^T$.

4. Escape rate

In this section we calculate the cumulative phase-space volume per unit of energy hypersurface which escapes in time τ for the circular case. A star can escape the cluster only if its energy E is above the critical energy E_c (see Figure 2). However there are family of stable orbits, known as potential escapers, with $E > E_c$ which do not escape the cluster (see Figure 3). Therefore when calculating the flux of escaping stars we need to exclude these orbits. One option is to assume that all stars with $E > E_c$ and $x^2 \geq x_L^2$ escape the cluster. Although this criterion is a necessary condition for escape, it does not exclude the small fraction of "fake escapers" orbits which escape through one of the Lagrange points but then being recaptured as shown in Figure 3. Nevertheless since the fraction of fake escaper is small we will follow Fukushige & Heggie (2000) and use this criterion. Under this definition, a star escapes by "time" τ if the value of $\xi(\tau) > 0$, and so the volume of phase space on the corresponding excess energy, $\epsilon = (E - E_c) / |E_c|$, hypersurface which has escaped by that time is given by

$$\mathcal{N}(\epsilon, \tau) = \int \Theta(\xi(\tau)) \, \delta\left(\epsilon - \epsilon\left(\xi, \tilde{y}, \xi', \tilde{y}'\right)\right) d\xi d\tilde{y} d\xi' d\tilde{y}', \tag{63}$$

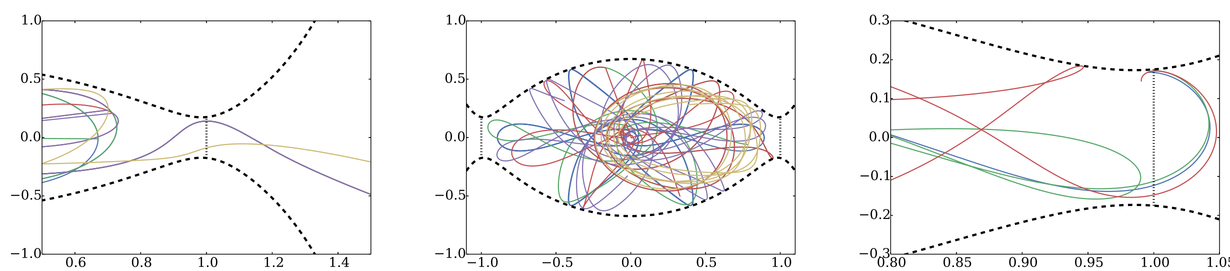

Fig. 3.— The characterization of a population of escapers in a star cluster is a subtle issue, and three main categories can be identified. Real escapers (left): stars with $E > E_c$ and $x > r_L$ for t larger then some t_0. Potential escapers (middle): Stars with $E > E_c$ but $x < r_L$ for all times. Fake escapers (right): Stars with $E > E_c$ but $x > r_L$ only for a finite period of time.

where the excess is

$$\epsilon = (E - E_c) / |E_c| = \frac{1}{9}\left(\xi'^2 + \tilde{y}'^2\right) - \xi^2 + \frac{1}{3}\tilde{y}^2, \tag{64}$$

and we used $\left|\tilde{E}_c\right| = 9/2$. The flux is given by

$$\begin{aligned} \mathcal{F}(\epsilon) &= \mathcal{N}'(\epsilon, 0) = 2\int_{\xi'>0} \delta\left(\epsilon - \epsilon\left(\xi = 0, \tilde{y}, \xi', \tilde{y}'\right)\right) \xi' d\tilde{y} d\xi' d\tilde{y}' \\ &= 27\sqrt{3}\epsilon \end{aligned} \tag{65}$$

It is convince to work in coordinates in which the excess energy hypersurface has a simple geometrical representation. Thus, we use the change of variables

$$\xi = -\frac{\chi\sqrt{\epsilon}}{\sqrt{1-\chi^2}};\ \tilde{y} = \frac{\sqrt{3\epsilon}}{\sqrt{1-\chi^2}} g_1 \tag{66}$$

$$\xi' = \frac{3\sqrt{\epsilon}}{\sqrt{1-\chi^2}} g_2;\ \tilde{y}' = \frac{3\sqrt{\epsilon}}{\sqrt{1-\chi^2}} g_3 \tag{67}$$

in which the excess energy hypersurface is given by the unit sphere $g_1^2 + g_2^2 + g_3^2 = 1$ and $0 < \chi < 1$. Thus,

$$\mathcal{N}(\epsilon, \tau) = 9\sqrt{3}\epsilon \int_0^1 d\chi \int d^3\mathbf{g} \frac{\Theta\left(\tilde{\mathbf{k}}(\tau) \cdot \mathbf{g} - \chi\right)}{\left(1-\chi^2\right)^2} \delta\left(1 - g^2\right) \tag{68}$$

where $\mathbf{g} = (g_1, g_2, g_3)$, and using equation (62) (with $e = 0$), we have

$$\xi(\tau) = \frac{\sqrt{\epsilon} k_0}{\sqrt{1-\chi^2}} \left(\tilde{\mathbf{k}}(\tau) \cdot \mathbf{g} - \chi\right), \tag{69}$$

where we defined

$$\tilde{\mathbf{k}}(\tau) = \left(\sqrt{3}k_1(\tau), 3k_2(\tau), 3k_3(\tau)\right) / k_0(\tau). \tag{70}$$

Since $\tilde{\mathbf{k}}(\tau) \cdot \mathbf{g} - \chi = 0$ defines a two dimensional plane, the integral over $\mathbf{g}$ in equation (68) represents the surface of the spherical cap above the plane $\tilde{\mathbf{k}}(\tau) \cdot \mathbf{g} - \chi = 0$ (see Figure 4). Therefore

$$\int d^3\mathbf{g}\Theta\left(\tilde{\mathbf{k}}(\tau) \cdot \mathbf{g} - \chi\right) \delta\left(1 - g^2\right) = 2\pi\left(1 - \chi/\tilde{k}(t)\right), \tag{71}$$

where $\tilde{k} = |\tilde{\mathbf{k}}|$, and

$$\mathcal{N}(\epsilon, \tau) = 18\sqrt{3}\pi\epsilon \int_0^{\tilde{k}(t)} \frac{1 - \chi/\tilde{k}(t)}{\left(1-\chi^2\right)^2} d\chi = 9\sqrt{3}\pi\epsilon \tanh^{-1}\left(\tilde{k}(\tau)\right). \tag{72}$$

It is easy to show that $\tilde{k}(\tau) < 1$ for finite τ and $\lim_{\tau\to\infty} \tilde{k}(\tau) = 1$, and therefore $\lim_{\tau\to\infty} \mathcal{N}(\epsilon, \tau) = \infty$ as expected.

If the escape rate was constant, then the cumulative number of escapers would be linear with time $\mathcal{N}(\epsilon, \tau) = F(\epsilon)\tau$. However, as shown in Figure 5 this is not the case and over one orbital period of the cluster around the galaxy the deviation is about 20%.

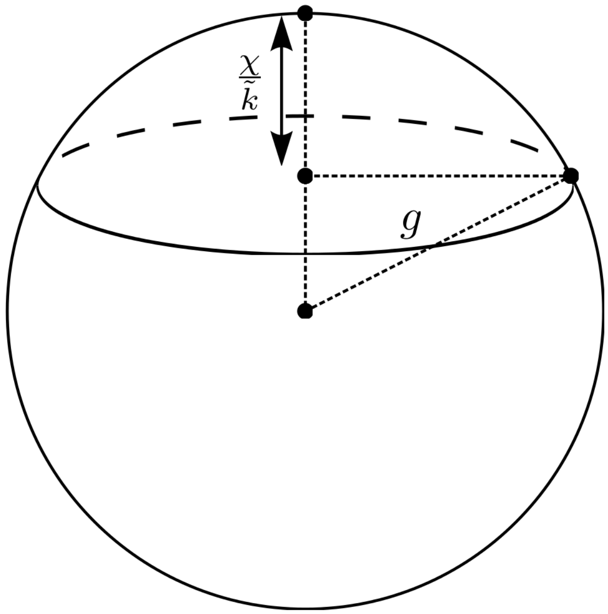

Fig. 4.— The spherical cap above the plane $\tilde{\mathbf{k}}\,(\tau) \cdot \mathbf{g} - \chi = 0$

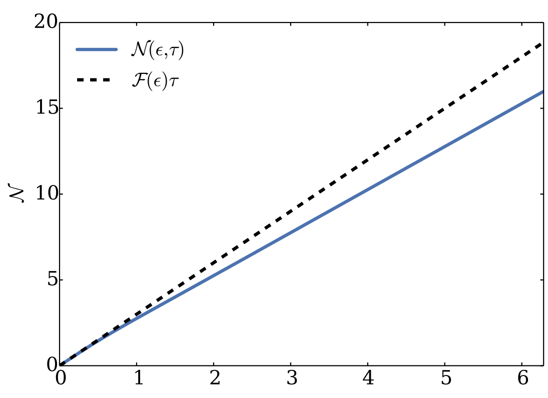

Fig. 5.— The cumulative phase-space volume per unit of energy hypersurface which escapes in time τ as a function of τ

5. Summary and future work

The goal of this project is to calculate the escape rate from clusters on elliptical orbits. We derived the equation of motion for general galactic and cluster potentials and for a general eccentricity of the cluster's orbit around the galaxy. We found unstable periodic orbits which are the generalization of the Lagrange points in the circular case. We rewrote the equations of motion in a regularized manner in which these unstable orbits are mapped into fixed points. We linearized these regularized equations near the Lagrange points to calculate the cumulative escaping volume per unit of energy as a function of time. The escape rate in the circular case is calculated based on the fact that the energy is conserved. This is not the case in the elliptical case. Therefore in the elliptical case the escaping volume is not a monotonic function as stars can be recaptured along the orbit as the effective potential changes. In the future we intend to apply these results to the elliptical case by calculating the escaping volume over one orbital time. This can be done by calculating the linearized equation of motion, which are now time-dependent, numerically or perturbatively. This will enable us to predict the number of stars escaping the cluster as a function of time which can then by compared to N-body simulations.

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