The elliptical Hill's problem and its application to escape from star clusters

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Basic Properties of Globular Clusters

- Nearly spherical
- About 150 globular clusters in the Milky Way.
- Age: $\gtrsim 10^{10}\,\mathrm{yr}$
- Median distance from Galactic Center: $\sim 9.3\,\mathrm{kpc}$
- Number of stars: $10^3 - 10^6$
- Median half-mass radius: $\sim 3.08\,\mathrm{pc}$
- Median tidal radius: $\sim 34.5\,\mathrm{pc}$
- Relaxation time: $\sim 10^9\,\mathrm{yr}$
- Meany on eccentric orbits

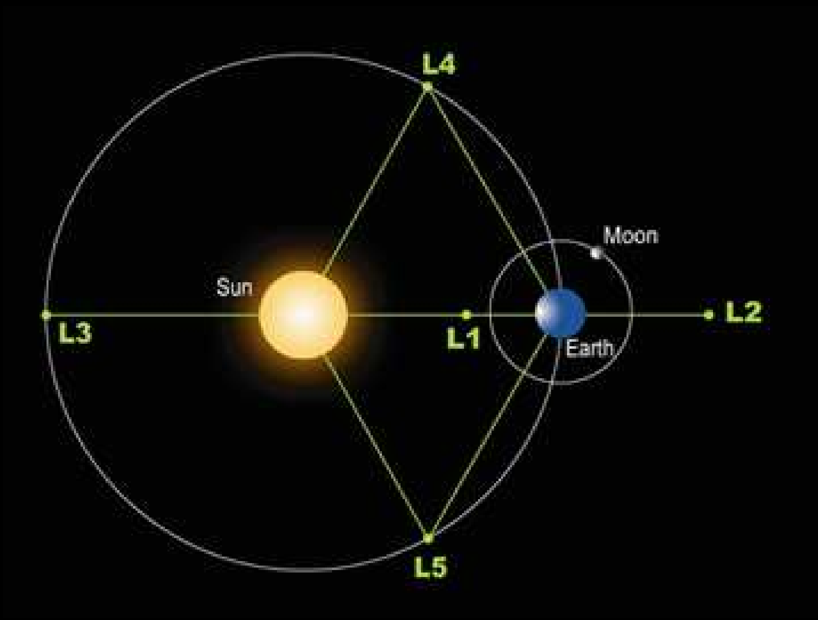
M4: The Closest Known Globular Cluster (AOPD 2000 May 23)

Modeling Globular Clusters

- Snapshot modeling:
  - Constructing a consistent, stationary, distribution function $f(v,x)$ that describe the quasi-equilibrium state of the cluster e.g. King model, Plummer model (see Kate's talk)
- Evolutionary modeling
  - N-body (See Maxwell and Filippo's talks)

Hill's Problem: Context

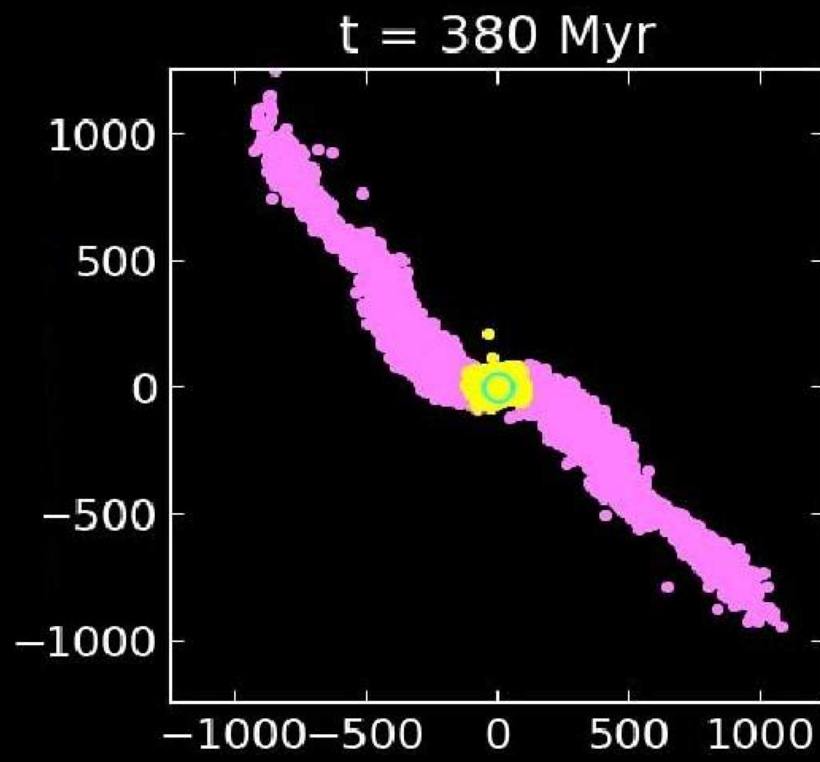
- The three body problem: Moon, Earth and Sun
- Applicable to nearly circular orbit
- Has many applications: (Solar system, Star clusters, binary disruption around a massive black hole, etc.)



From the U.S. Navy web site

Application to escape from star cluster

- Due to the tidal forces stars gradually escape from the cluster
- Expected on theoretical grounds and confirmed by observations (Stephen's talk) and simulations (Maxwell and Filippo's talks)
- The energy of stars changes by two-body encounters
- If the stellar energy exceeds some critical energy the star can escape

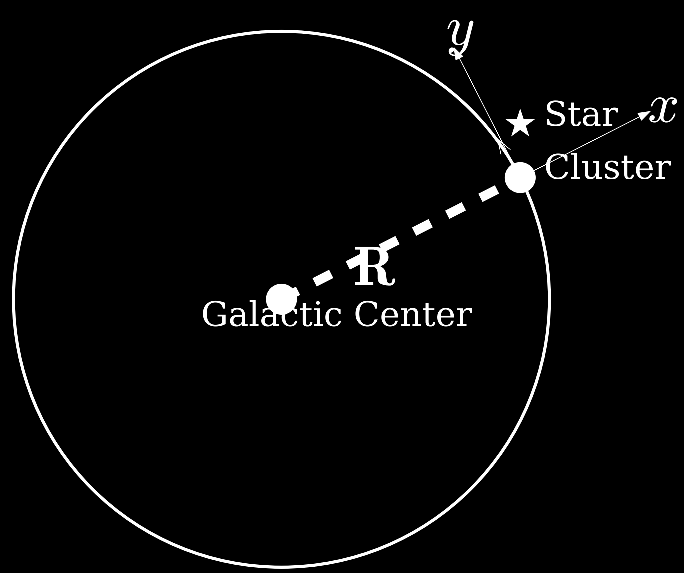


See Filippo's talk

Cluster on a circular orbit

- Position vector: $\mathbf{x} = \mathbf{R} + \mathbf{r}$
- Rotation: $\boldsymbol{\Omega} = (\Omega, 0, 0)$
- Velocity: $\dot{\mathbf{x}} = \boldsymbol{\Omega} \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{r}}$, where $\dot{\mathbf{r}}$ is the velocity in the rotating frame
- Acceleration:

$$\begin{aligned} \ddot{\mathbf{x}} &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\mathbf{R} + \mathbf{r})) + 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} + \ddot{\mathbf{r}} \\ &= -\nabla\Phi_g(\mathbf{R} + \mathbf{r}) - \nabla\Phi_c(\mathbf{r}) \end{aligned}$$



- Equations of motion in the rotating frame: Centrifugal, Coriolis and gravitational forces

$$\ddot{\mathbf{r}} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times (\mathbf{R} + \mathbf{r})) - 2\boldsymbol{\Omega} \times \dot{\mathbf{r}} - \nabla\Phi_g(\mathbf{R} + \mathbf{r}) - \nabla\Phi_c(\mathbf{r})$$

The tidal approximation

- Assuming $r \ll R$ and a spherical symmetric potential:

$$\nabla\Phi_g(\mathbf{r}+\mathbf{R}) \quad = \quad \Phi_g'(R)\hat{R} + (\mathbf{r}\cdot\nabla)\Phi_g'(\mathbf{R})\hat{R}$$

- Equations of motion

$$\ddot{\mathbf{r}} \quad = \quad -2\Omega\times\dot{\mathbf{r}} - \nabla\Phi_{eff}(\mathbf{r})$$

- Effective Potential

$$\Phi_{eff}(\mathbf{r}) = \frac{1}{2}\Omega^2 z^2 + \frac{1}{2}\left(\kappa^2 - 4\Omega^2\right)x^2 + \Phi_c(\mathbf{r})$$

where $\kappa^2 = \Phi_g''(R) + 3\Omega^2$

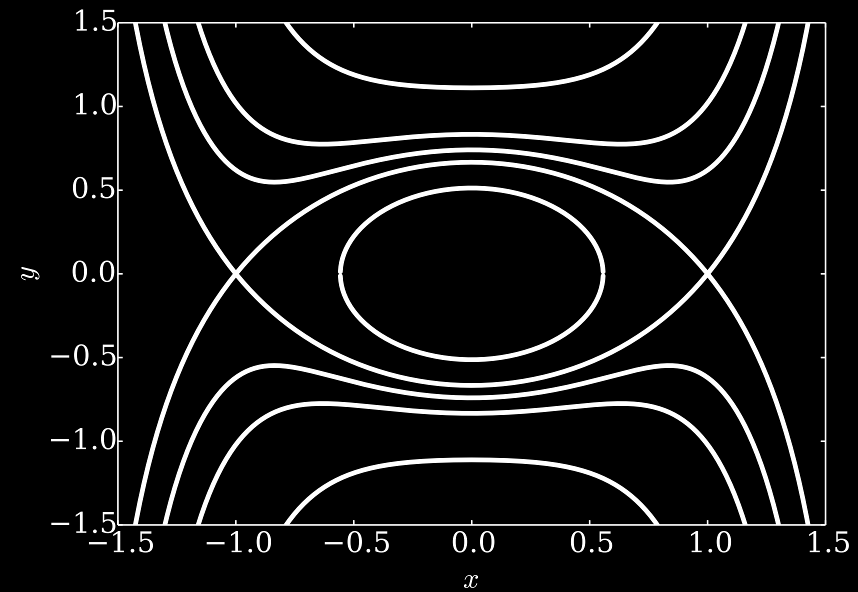
The tidal radius

- The energy is an integral of motion (the Jacobi integral)

$$E = \frac{1}{2}\dot{r}^2 + \Phi_{eff}$$

- Lagrange points: Stationary orbit of the form $\mathbf{r} = (\pm r_t, 0, 0)$.

$$r_t = -\Phi_c'(r_t) / \left(\kappa^2 - 4\Omega^2\right)$$



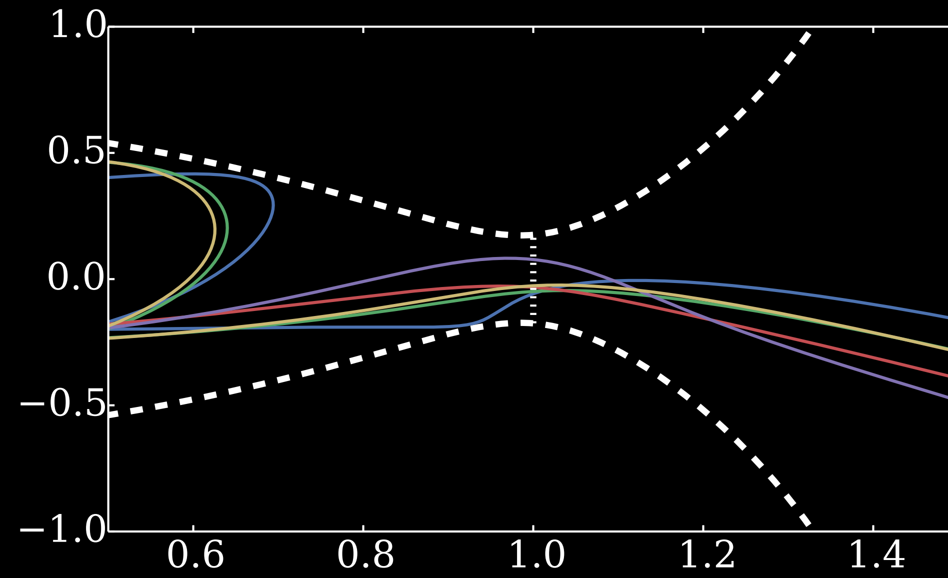
- The critical energy

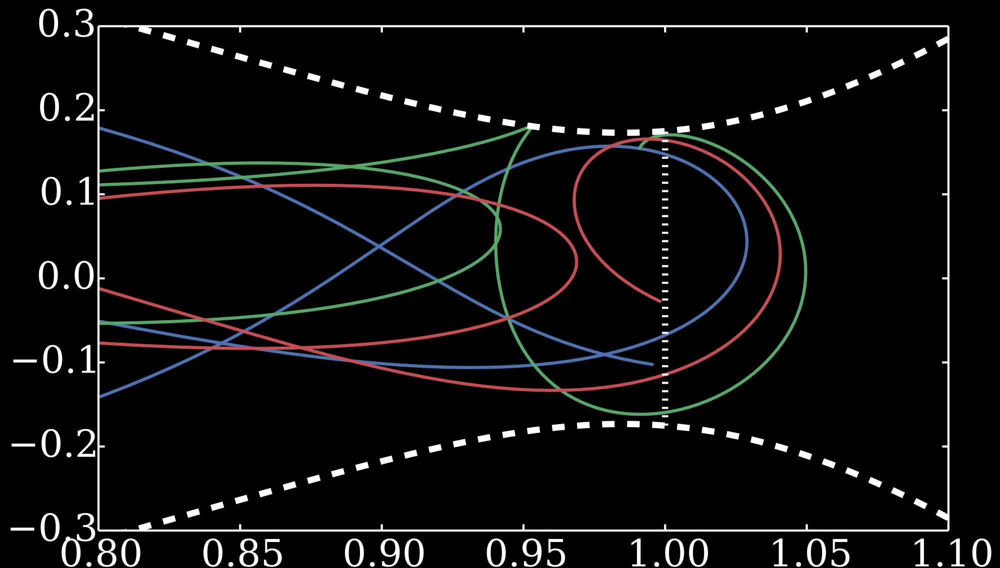
$$E_c = E(r_t) = \frac{3}{2}\Phi_c(r_t)\left(1 - \frac{1}{3}\frac{d\log r\Phi_c(r)}{d\log r}\bigg|_{r=r_t}\right)$$

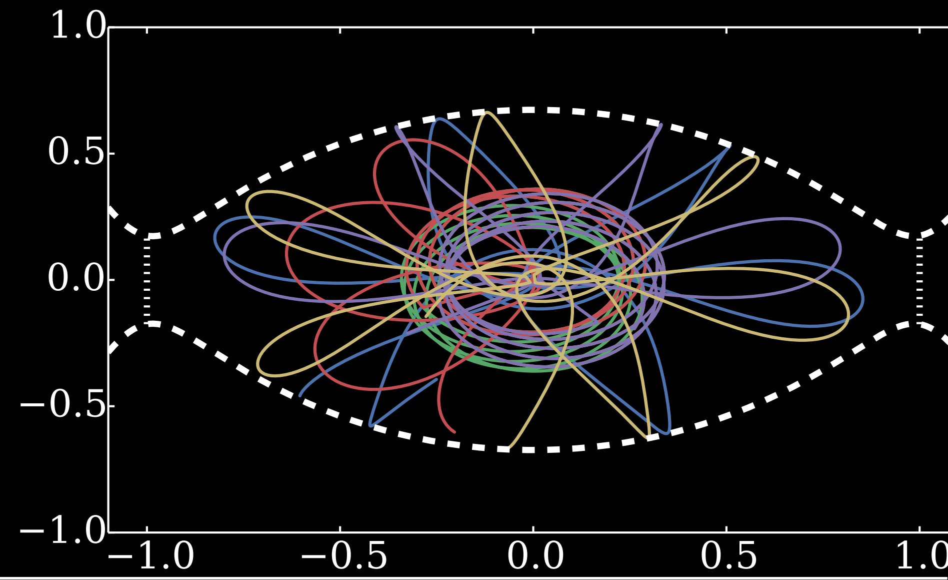
- Assuming a Keplerian potential $\Phi_c(r) = -GM/r$; $r_t^3 = \frac{GM}{4\Omega^2-\kappa^2}$; $E_c = -\frac{3}{2}\frac{GM}{r_t}$

Escaping from the cluster

- Real escapers: $E > E_c$ and $x(\infty) > r_t$
- Potential escapers (see Kate's Talk): $E > E_c$ but $x(t) < r_t$ for all t.
- Fake escapers: $E > E_c$ and $x(t) > r_t$ for some t but $x(\infty) < r_t$.



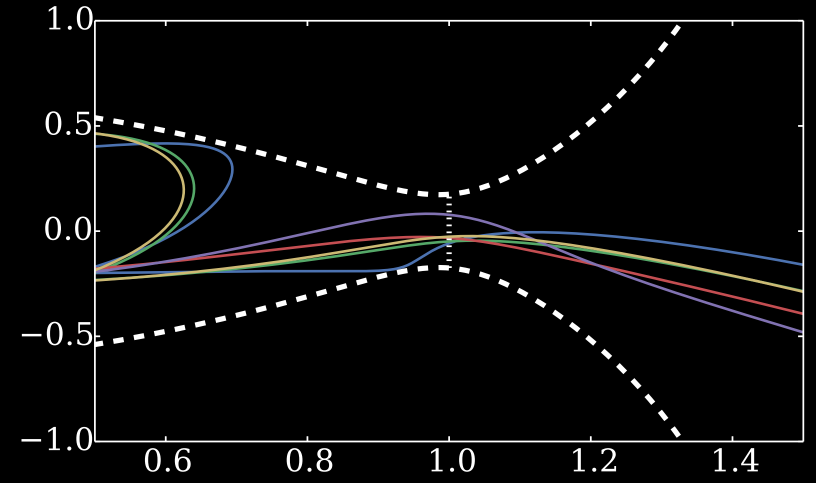




Escape rate

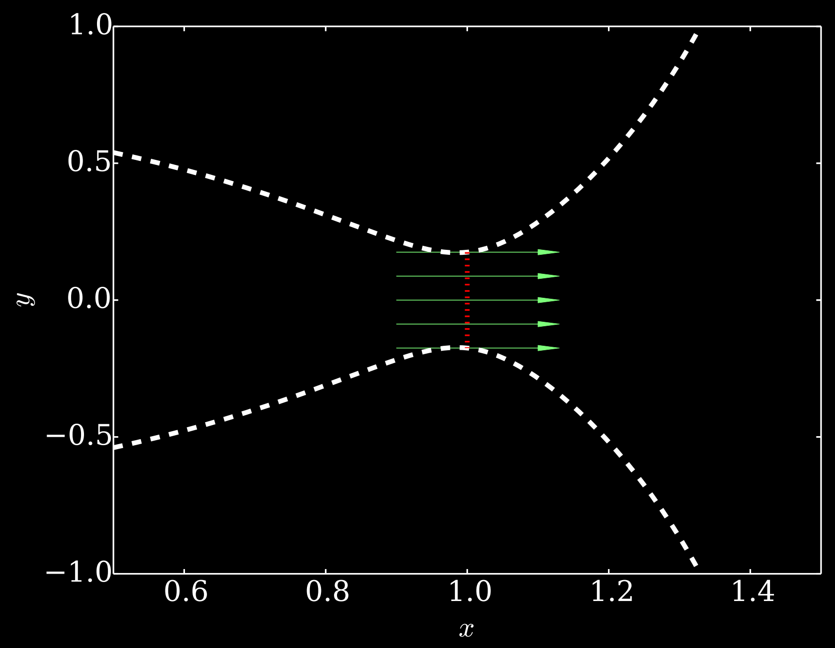
- Flux: The phase volume, per unit energy, across $x = r_t$ (Fukushige Heggie 2000)

$$\mathcal{F}(\varepsilon) = \int_{\dot{x}>0} \delta\left(E - E(\mathbf{r}, \dot{\mathbf{r}})\right) \dot{x} d^3\dot{\mathbf{r}} dy dz$$



- Volume: phase space volume per unit of energy

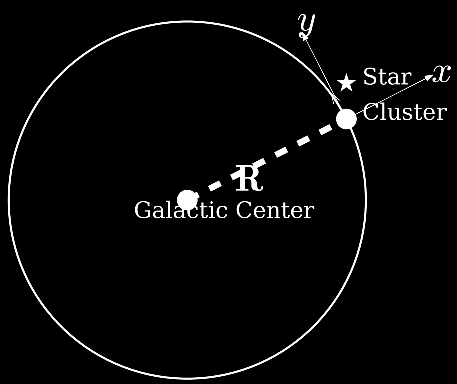
$$\mathcal{V}(\varepsilon) = \int \delta\left(E - E(\mathbf{r}, \dot{\mathbf{r}})\right) d^3\mathbf{r} d^3\dot{\mathbf{r}}$$

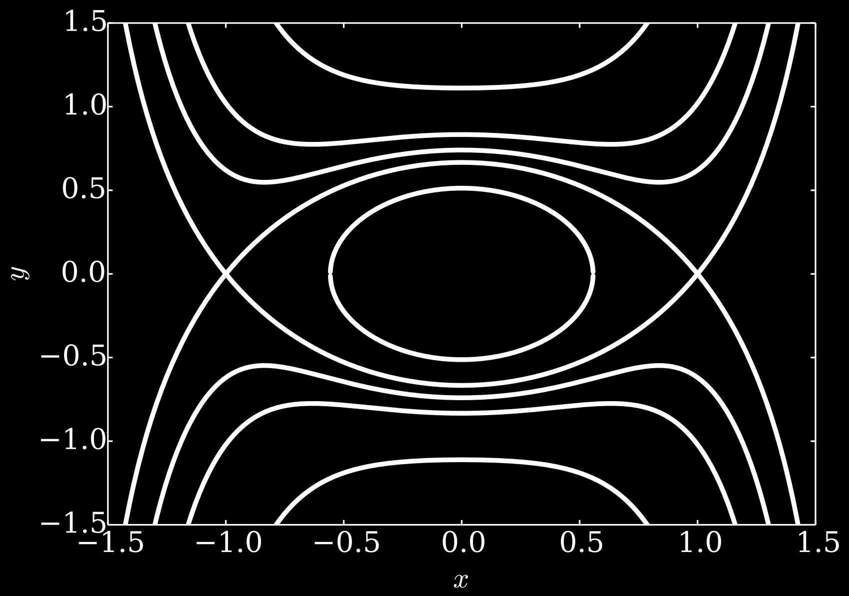


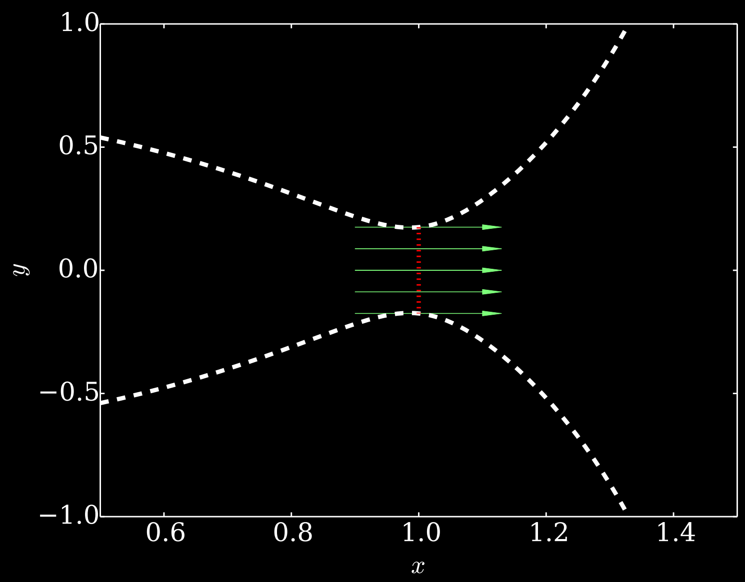
- Escape time $t_e = \frac{\mathcal{V}}{\mathcal{F}}$

Circular case: Summary

- Integral of motion: The Jacobi integral
- Lagrange points: unstable stationary points
- Critical energy: Escape criterion
- Escape Flux

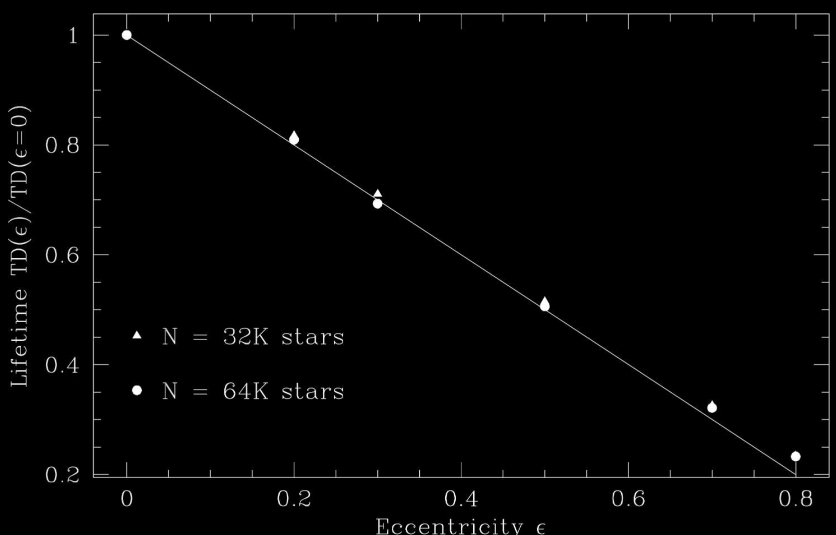




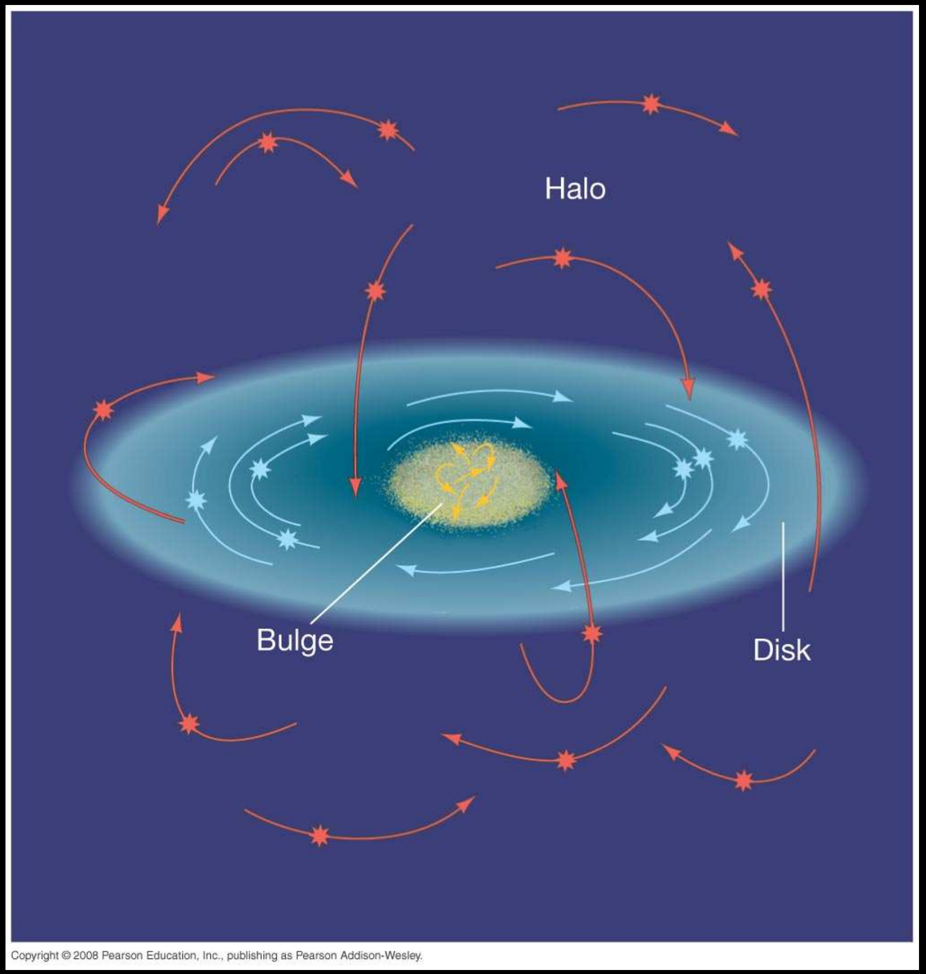


Cluster on eccentric orbit: Motivation

- Stars escape the cluster due to the galactic tidal filed
- Most of the GCs are on eccentric orbits
- The evolution of the cluster depends on the eccentricity (See Maxwell and Filippo's talks)
- No theory for estimating the escape rate from clusters on eccentric orbits

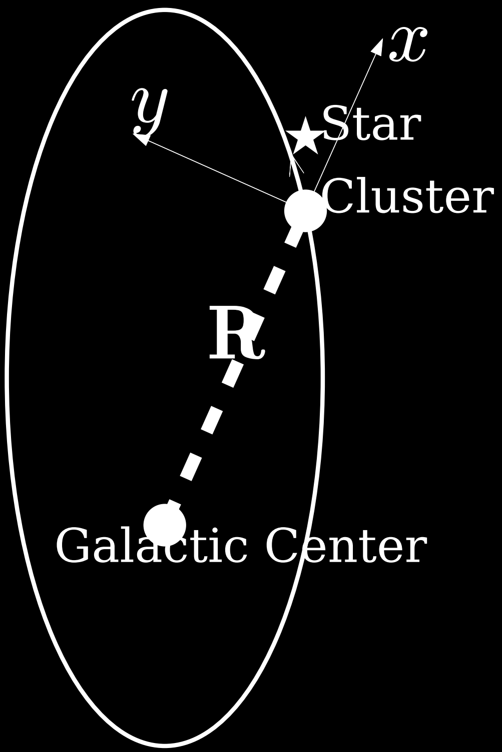


Baumgardt & Makino 2003



Cluster on eccentric orbit: Challenges

- No integral of motion
- No stationary orbits
- Phase dependent critical energy
- Phase dependent flux



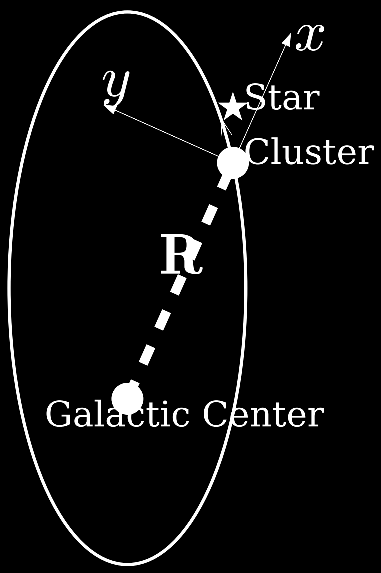
Outline of the Analytical approach

- Generalized Equations of motion for a cluster on an elliptic orbit
- Unstable periodic orbits: generalization of the Lagrange points
- Three-body toy model: star, cluster, galaxy
- Linearized equations of motion near the “Lagrange points”
- Escape volume over one orbital time

Cluster on an elliptic orbit

- Position vector: $\mathbf{x} = \mathbf{R} + \mathbf{r}$
- Rotation: $\Omega = (\Omega, 0, 0)$
- Velocity: $\dot{\mathbf{x}} = \Omega \times (\mathbf{R} + \mathbf{r}) + \dot{\mathbf{R}} + \dot{\mathbf{r}}$, where $\dot{\mathbf{r}}$ are $\dot{\mathbf{R}}$ are the velocities in the rotating frame
- Acceleration:

$$\begin{aligned} \ddot{\mathbf{x}} &= \Omega \times (\Omega \times (\mathbf{R} + \mathbf{r})) + \dot{\Omega} \times (\mathbf{R} + \mathbf{r}) + \ddot{\mathbf{R}} + 2\Omega \times \left(\dot{\mathbf{R}} + \dot{\mathbf{r}}\right) + \ddot{\mathbf{r}} \\ &= -\nabla\Phi_g (\mathbf{R} + \mathbf{r}) - \nabla\Phi_c (\mathbf{r}) \end{aligned}$$



- Equations of motion in the rotating frame: Centrifugal, Coriolis and gravitational forces

$$\ddot{\mathbf{r}} = -\Omega \times (\Omega \times (\mathbf{R} + \mathbf{r})) - 2\Omega \times \dot{\mathbf{r}} - \Omega \times \dot{\mathbf{R}} - \dot{\Omega} \times (\mathbf{R} + \mathbf{r}) - \ddot{\mathbf{R}} - \nabla\Phi_g (\mathbf{R} + \mathbf{r}) - \nabla\Phi_c (\mathbf{r})$$

Generalized equation of motion

- Equation of motion in the tidal approximation

$$\begin{aligned}\ddot{\mathbf{r}} &= -2(\mathbf{r}\cdot\Omega)\Omega - \left(\mathbf{r}\cdot\hat{R}\right)\left(\kappa^2 - 4\Omega^2\right)\hat{R} \\ &\quad +2\frac{\dot{R}}{R}\Omega\times\mathbf{r} + \frac{\ddot{R}}{R}\left(\mathbf{r} - x\hat{R}\right) \\ &\quad -2\Omega\times\dot{\mathbf{r}} - \nabla\Phi_c(\mathbf{r})\end{aligned}$$

where Ω, κ and R are time dependent.

- Generalized tidal radius: Periodic orbits of the form $\mathbf{r} = (\pm r_t(t), 0, 0)$

$$\begin{aligned}\ddot{x} &= -r_t\left(\kappa^2 - 4\Omega^2\right) - \Phi_c'(r_t) \\ \ddot{y} &= 2\frac{\Omega}{R}\left(\dot{R}r_t - R\dot{r}_t\right) = 0\end{aligned}$$

- Therefore $r_t = \alpha R$ and $\ddot{r}_t = \alpha\ddot{R}$. A solution exist if we can solve

$$\Phi_c'(\alpha R)/\alpha = \Phi_g'(R) - R\Phi_g''(R)$$

for constant α.

Regularization

- Consider the transformation $\tilde{\mathbf{r}} = \mathbf{r}/r_t$, $\tau = \Omega t$.
- Equations of motion:

$$\ddot{\tilde{\mathbf{r}}} = -\nabla_{\tilde{\mathbf{r}}}\tilde{\Phi}_{eff}\left(\tilde{\mathbf{r}}\right) - 2\hat{z}\times\dot{\tilde{\mathbf{r}}}$$

with

$$\tilde{\Phi}_{eff} = \frac{1}{2}\tilde{z}^2 - \frac{1}{2}\beta\tilde{x}^2 + \tilde{\Phi}_c\left(\tilde{r}\right)$$

where $\tilde{\Phi}_c\left(\tilde{r}\right) = r_t^{-2}\Omega^{-2}\Phi_c\left(\tilde{r}r_t\right)$ and $\beta = \tilde{\Phi}_c\left(1\right)\frac{d\log\Phi_c(r_t)}{d\log r_t}$ are phase dependent.

- Stationary 'Lagrange points' $\tilde{x} = \pm 1$
- Energy: $\tilde{E} = \frac{1}{2}\dot{\tilde{r}}^2 + \tilde{\Phi}_{eff}$
- Phase dependent critical energy:

$$\tilde{E}_c = \frac{3}{2}\Omega^{-2}r_t^{-2}\Phi_c\left(r_t\right)\left[1 - \frac{1}{3}\left.\frac{d\log r\Phi_c\left(r\right)}{d\log r}\right|_{r=r_t}\right]$$

Keplerian Potential

- Assuming the cluster potential is Keplerian, $\Phi_c(r_t) = -GM/r$ there is an unstable periodic orbit if the galactic potential is of the form

$$\Phi_g(R) = -\frac{GM_g}{R} + c_1 R^2 + c_2$$

and

$$r_t^3 = \frac{1}{3}\frac{M}{M_g}R^3$$

- Critical energy

$$\tilde{E}_c = -\frac{9}{2}\frac{GM_g}{R^3\Omega^2} = -\frac{9}{2}\frac{R}{L^2}GM_g$$

- Equation of motion

$$\ddot{\tilde{\mathbf{r}}} = -\nabla_{\tilde{\mathbf{r}}}\tilde{\Phi}_{eff}(\tilde{\mathbf{r}}) - 2\hat{z}\times\dot{\tilde{\mathbf{r}}}$$

with

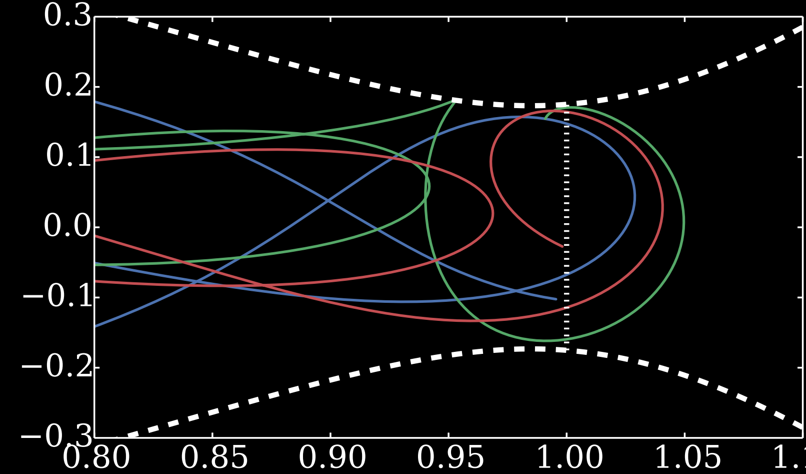
$$\tilde{\Phi}_{eff} = \frac{1}{2}\tilde{z}^2 - \frac{2}{3}\left|\tilde{E}_c\right|\left(\frac{1}{\tilde{r}} + \frac{1}{2}\tilde{x}^2\right)$$

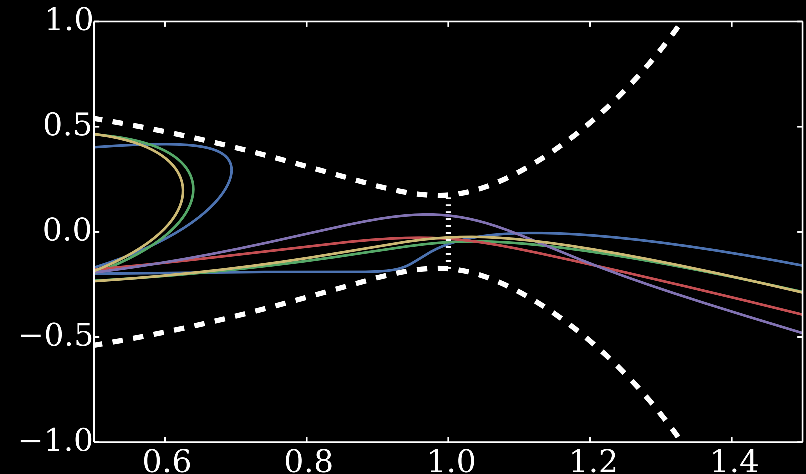
Linearizion

- Assuming Keplerian potential
  $\Phi_c(r) = -GM/r$
- Expansion near the Lagrange point $\xi = x - 1$
- Effective potential $\tilde{\Phi}_{eff} \approx \tilde{E}_c \left(1 + \xi^2 - \frac{1}{3}\tilde{y}^2\right)$
- Linearized equations of motion

$$\begin{aligned} \ddot{\xi} &= 2|E_c|\xi + 2\dot{y} \\ \ddot{y} &= -\frac{2}{3}|E_c|\xi - 2\dot{\xi} \end{aligned}$$

- For the circular case $\tilde{E}_c$ is constant
- Eigenvalues
  — Real: $\pm\sqrt{1+2\sqrt{7}} \approx \pm 2.5$
  — Imaginary: $\pm i\sqrt{2\sqrt{7}-1} \approx \pm 2.07i$





Escape rate

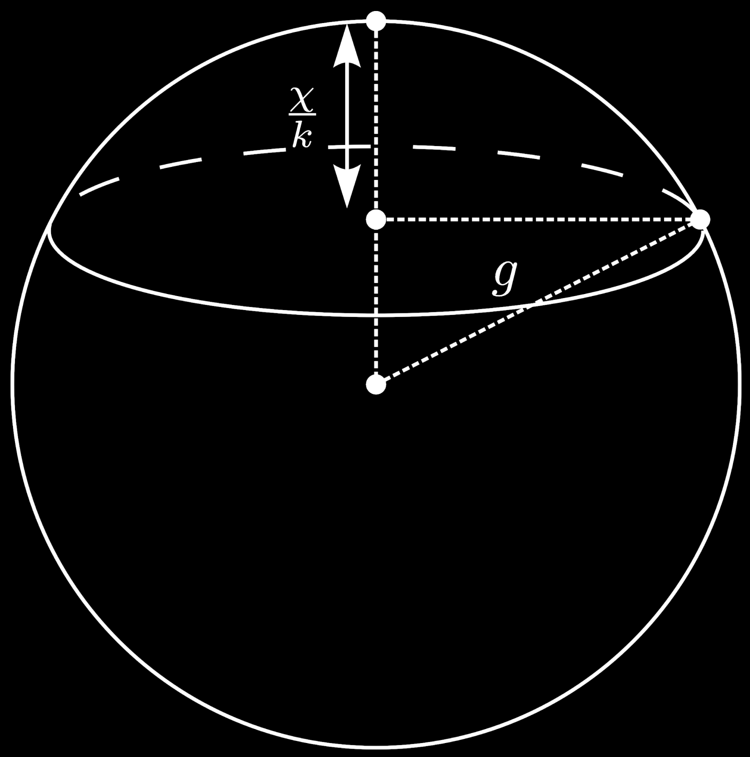
- Escape volume per unit of energy $\varepsilon = (E - E_c) / |E_c|$ at time τ

$$\mathcal{N}(\varepsilon,\tau) = \int \Theta(\xi(\tau)) \left|\tilde{E}_c\right|^{-1} \delta\left(\varepsilon - \frac{1}{2}\left|\tilde{E}_c\right|^{-1}\left(\dot{\xi}^2 + \dot{\tilde{y}}^2\right) + \left(\xi^2 - \frac{1}{3}\tilde{y}^2\right)\right) d\xi d\tilde{y} d\dot{\xi} d\dot{\tilde{y}}$$

- Change of variables

$$\tilde{x} = -\frac{\sqrt{\varepsilon}\chi}{\sqrt{1-\chi^2}}; \;\; \dot{\tilde{x}} = \frac{\sqrt{3\varepsilon}}{\sqrt{1-\chi^2}} g_2$$

$$\tilde{y} = \frac{3\sqrt{\varepsilon}}{\sqrt{1-\chi^2}} g_1; \;\; \dot{\tilde{y}} = \frac{3\sqrt{\varepsilon}}{\sqrt{1-\chi^2}} g_3$$



Then

$$\mathcal{N}(\varepsilon,\tau) = 2\sqrt{3}\varepsilon \int \frac{\Theta(\mathbf{k}(\tau)\cdot\mathbf{g} - \chi)}{\left(1-\chi^2\right)^2} \delta\left(g^2 - 1\right) d\chi d^3\mathbf{g}$$

where $\xi(\tau) \propto \mathbf{k}(\tau)\cdot\mathbf{g} - \chi$.

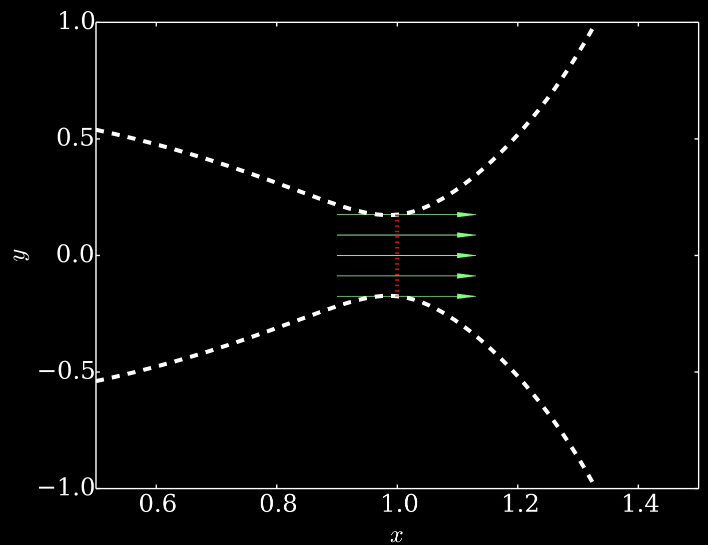
Time dependent flux

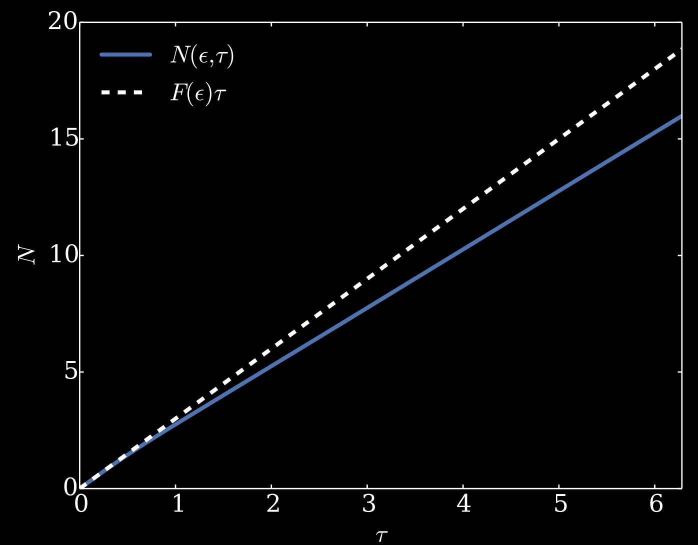
- Escape volume per unit of energy $\varepsilon = (E - E_c)/|E_c|$ at time τ

$$\mathcal{N}(\varepsilon,\tau) \quad = \quad \sqrt{3}\varepsilon \int_0^k \frac{2\pi(1-\chi/k(\tau))}{(1-\chi^2)^2} d\chi = \sqrt{3}\pi\varepsilon \tanh^{-1}(k(\tau))$$

- Flux

$$\mathcal{F}(\varepsilon) = \dot{\mathcal{N}}(\varepsilon,0) = \left|\tilde{E}_c\right|^{-1} \int_{\dot{\xi}>0} \delta\left(\varepsilon - \varepsilon\left(\xi,\dot{\xi},\tilde{y},\dot{\tilde{y}}\right)\right) \dot{\xi} d\tilde{y} d\dot{\xi} d\dot{\tilde{y}} = 3\sqrt{3}\pi\varepsilon$$





The elliptical case

Elliptical case

- Phase dependent potential

$$\tilde{\Phi}_{eff} \approx -|E_c| \left(1 + \xi^2 - \frac{1}{3}\tilde{y}^2\right)$$

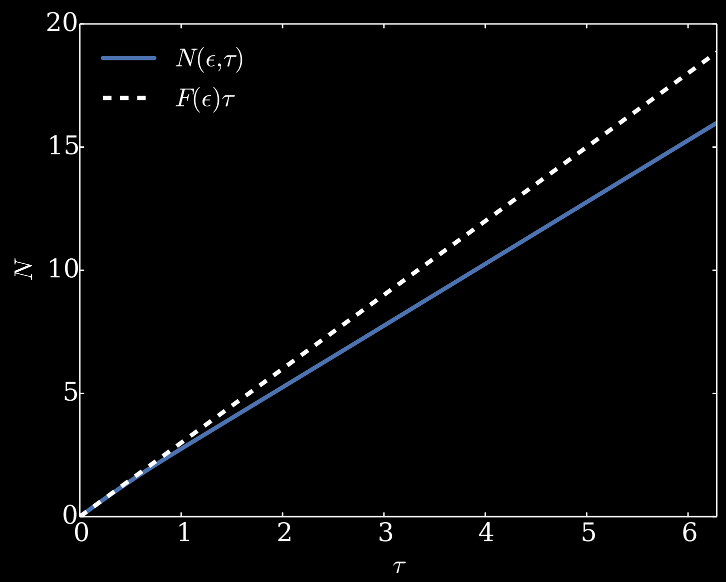
with

$$|E_c| = -\frac{9}{2}\left[1 + e\cos\left(\psi_0 + \tau\right)\right]$$

- Flux $\tilde{\mathcal{F}}\left(\varepsilon\right) = \mathcal{N}\left(\varepsilon, 2\pi\right)/2\pi$
- Escape volume per unit of energy over one orbital time $\varepsilon = \left(E - E_c\right)/|E_c|$

$$\mathcal{N}\left(\varepsilon, 2\pi\right) = \int \frac{\Theta\left(\xi\left(2\pi\right)\right)}{|E_c\left(\tau = 0\right)|} \delta\left(\varepsilon - \varepsilon\left(\xi, y, \dot{\xi}, \dot{y}, \tau = 0\right)\right) d\xi d\tilde{y} d\dot{\xi} d\dot{\tilde{y}}$$

Circular case



- Flux $\mathcal{F}\left(\varepsilon\right) = \dot{\mathcal{N}}\left(\varepsilon, 0\right)$

Future agenda

- Solving the time-dependent linear equations of motion
  - Numerically
  - Analytically: Perturbative approach
- Calculating escape volume per unit of energy over one orbital time
- Excluding fake escapers
- Comparison with numerical simulations (see Maxwell and Filippo's talk)